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FEKETE-SZEGÖ INEQUALITY FOR FUNCTIONS BELONGING TO A CERTAIN CLASS OF ANALYTIC FUNCTIONS INTRODUCED USING LINEAR COMBINATION OF VARIATIONAL POWERS OF STARLIKE AND CONVEX FUNCTIONS

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Abstract

We introduce some classes of analytic functions, its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

Keywords: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

1. Introduction

M Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([1], [2]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [10] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and $a_2{}^2$ for the class $\mathcal{S}$,Fekete and Szegö[4] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases} \qquad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$([3], [9]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

$\in \mathcal{A}$ and satisfying the condition

$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$

(1.3)

We denote by $\mathcal{K}$, the class of univalent convex functions

$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \boldsymbol{\mathcal{A}}$ and satisfying the condition

$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}.$

(1.4)

A function $f(z) \in \boldsymbol{\mathcal{A}}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$Re\left(\frac{zf\prime(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$

(1.5)

The class of close to convex functions is denoted by C and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$S^*(A,B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$

(1.6)

$\mathcal{K}(A,B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{(zf'(z))\prime}{f\prime(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$

(1.7)

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\ (A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as $\left\{f(z) \in \boldsymbol{\mathcal{A}}; \alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\beta} + (1-\alpha)\left(\frac{(zf'(z))'}{f\prime(z)}\right)^{\beta} \prec \frac{1+z}{1-z}; z \in \mathbb{E}\right\}$ and we will denote this class as $S^*(f,f',\alpha,\beta)$.

We will deal with two subclasses of $S^*(f,f',\alpha,\beta)$ defined as follows in our next paper:

$S^*(f,f',\alpha,\beta,A,B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\beta} + (1-\alpha)\left(\frac{(zf'(z))'}{f\prime(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz}; z \in \mathbb{E}\right\}$ (1.8)

$S^*(f,f',\alpha,\beta,\delta) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\beta} + (1-\alpha)\left(\frac{(zf'(z))'}{f\prime(z)}\right)^{\beta} \prec \left(\frac{1+z}{1-z}\right)^{\mu}; z \in \mathbb{E}\right\}$(1.9)

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$

(1.10)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2.$

(1.11)

2. PRELIMINARY LEMMAS:

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots.$$

(2.1)

3. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in S^*(f, f', \alpha, \beta).$, then

$$|a_3 - \delta a_2^2| \leq \begin{cases} \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left[\frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)} - 4\mu\right], if \mu \leq \frac{4-6\alpha+12\beta+4\alpha^2-4\beta^2+3\alpha\beta^2-13\alpha\beta}{4(\alpha+3\beta-4\alpha\beta)}; & (3.1) \\ \frac{1}{\alpha+3\beta-4\alpha\beta} if \frac{4-6\alpha+12\beta+4\alpha^2-4\beta^2+3\alpha\beta^2-13\alpha\beta}{4(\alpha+3\beta-4\alpha\beta)} \leq \mu \leq \frac{37\alpha+38\beta-16\alpha^2-5\beta^2+24\alpha^2\beta+6\alpha\beta^2-67\alpha\beta-17}{4(\alpha+3\beta-4\alpha\beta)}; & (3.2) \\ \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left[4\mu - \frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)}\right], if\ \mu \geq \frac{37\alpha+38\beta-16\alpha^2-5\beta^2+24\alpha^2\beta+6\alpha\beta^2-67\alpha\beta-17}{4(\alpha+3\beta-4\alpha\beta)} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $S^*(f, f', \alpha, \beta)$, we have

$$\alpha\left(\frac{zf'(z)}{f(z)}\right)^{1-\beta} + (1-\alpha)\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)^{\beta} = \frac{1+w(z)}{1-w(z)}; w(z) \in \mathcal{U}.(3.4)$$

Expanding the series (3.4), we get

$$\alpha\left\{1 + (1-\beta)a_2 z + (2(1-\beta)a_3 - \frac{(1-\beta)(\beta+2)}{2}a_2^2)z^2 + - - -\right\} + (1-\alpha)\{1 + 2\beta a_2 z + 2\beta(3a_3 - (3-\beta)a_2^2)z^2 + - - -\} = (1 + 2c_1 z + 2(c_2 + {c_1}^2)z^2 + - - -). \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{2}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}}c_1$$

(3.6)

$$a_3 = \frac{1}{\alpha+3\beta-4\alpha\beta}c_2 + \frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}{c_1}^2.$$

(3.7)

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{1}{\alpha+3\beta-4\alpha\beta}c_2 + \left[\frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2} - \frac{4}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\mu\right]c_1^2.(3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{1}{(\alpha+3\beta-4\alpha\beta)}|c_2| + \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left|\frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)} - 4\mu\right||c_1^2|. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{(\alpha+3\beta-4\alpha\beta)}(1-|c_1|^2) + \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left|\frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha}{(\alpha+3\beta-4} - 4\mu\right||c_1^2$$

$$= \frac{1}{(\alpha+3\beta-4\alpha\beta)} + \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left[\left|\frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)} - 4\mu\right| - \frac{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}{(\alpha+3\beta-4\alpha\beta)}\right]|c_1|^2. \quad (3.10)$$

Case I: $\mu \leq \frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{4(\alpha+3\beta-4\alpha\beta)}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2 \leq \frac{1}{(\alpha+3\beta-4\alpha\beta)} + \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left[\frac{4-6\alpha+12\beta+4\alpha^2-4\beta^2+3\alpha\beta^2-13\alpha\beta}{(\alpha+3\beta-4\alpha\beta)} - 4\mu\right]|c_1|^2.(3.11)$$

Subcase I (a): $\mu \leq \frac{4-6\alpha+12\beta+4\alpha^2-4\beta^2+3\alpha\beta^2-13\alpha\beta}{4(\alpha+3\beta-4\alpha\beta)}$. Using (1.9), (3.11) becomes

$$|a_3-\mu a_2^2 \le \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left[\frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)}-4\mu\right]. \quad (3.12)$$

Subcase I (b): $\mu \ge \frac{4-6\alpha+12\beta+4\alpha^2-4\beta^2+3\alpha\beta^2-13\alpha\beta}{4(\alpha+3\beta-4\alpha\beta)}$. We obtain from (3.11)

$$|a_3-\mu a_2^2 \le \frac{1}{(\alpha+3\beta-4\alpha\beta)}. \quad (3.13)$$

Case II: $\mu \ge \frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{4(\alpha+3\beta-4\alpha\beta)}$

Preceding as in case I, we get

$$|a_3-\mu a_2^2 \le \frac{1}{(\alpha+3\beta-4\alpha\beta)}+\frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left[4\mu-\frac{37\alpha+38\beta-16\alpha^2-5\beta^2+24\alpha^2\beta+6\alpha\beta^2-67\alpha\beta-17}{(\alpha+3\beta-4\alpha\beta)}\right]|c_1|^2. \quad (3.14)$$

Subcase II (a): $\mu \le \frac{37\alpha+38\beta-16\alpha^2-5\beta^2+24\alpha^2\beta+6\alpha\beta^2-67\alpha\beta-17}{4(\alpha+3\beta-4\alpha\beta)}$

(3.14) takes the form $|a_3-\mu a_2^2 \le \frac{1}{(\alpha+3\beta-4\alpha\beta)}$ (3.15)

Combining subcase I (b) and subcase II (a), we obtain

$$\left|a_3-\mu a_2^2\right| \le \frac{1}{(\alpha+3\beta-4\alpha\beta)}\, if\, \frac{4-6\alpha+12\beta+4\alpha^2-4\beta^2+3\alpha\beta^2-13\alpha\beta}{4(\alpha+3\beta-4\alpha\beta)} \le \mu \le \frac{37\alpha+38\beta-16\alpha^2-5\beta^2+24\alpha^2\beta+6\alpha\beta^2-67\alpha\beta-17}{4(\alpha+3\beta-4\alpha\beta)} \quad (3.16)$$

Subcase II (b): $\mu \ge \frac{37\alpha+38\beta-16\alpha^2-5\beta^2+24\alpha^2\beta+6\alpha\beta^2-67\alpha\beta-17}{4(\alpha+3\beta-4\alpha\beta)}$

Preceding as in subcase I (a), we get

$$\left|a_3-\mu a_2^2\right| \le \frac{1}{\{\alpha(1-\beta)+2\beta(1-\alpha)\}^2}\left[4\mu-\frac{24\alpha+34\beta-8\alpha^2+12\alpha^2\beta+9\alpha\beta^2-49\alpha\beta-9\beta^2-13}{(\alpha+3\beta-4\alpha\beta)}\right]. \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z)=(1+az)^b$$

Where $a=\frac{\{(\alpha+2\beta-3\alpha\beta)^2+\alpha(1-\beta)(\beta+2)+4(1-\alpha)\beta(3-\beta)\}a_2^2-4(\alpha+3\beta-4\alpha\beta)a_3}{(\alpha+2\beta-3\alpha\beta)a_2}$

And $b=\frac{(\alpha+2\beta-3\alpha\beta)^2a_2^2}{\{(\alpha+2\beta-3\alpha\beta)^2+\alpha(1-\beta)(\beta+2)+4(1-\alpha)\beta(3-\beta)\}a_2^2-4(\alpha+3\beta-4\alpha\beta)a_3}$

Extremal function for (3.2) is defined by $f_2(z)=z(1+Bz^2)^{\frac{A-B}{2B}}$.

Corollary 3.2: Putting $\alpha=0, \beta=1$ in the theorem, we get

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 1-\mu, if \mu \le 1; \\ \frac{1}{3} if 1 \le \mu \le \frac{4}{3}; \\ \mu-1, if\ \mu \ge \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 3.3: Putting $\alpha=1, \beta=0$ in the theorem, we get

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 3-4\mu, if \mu \le \frac{1}{2}; \\ 1 if \frac{1}{2} \le \mu \le 1; \\ 4\mu-3, if\ \mu \ge 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

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