

Weighted Weibull-G Family of Distributions: Theory & Application in the Analysis of Renewable Energy Sources

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Abstract

Weighted distributions provide a unified approach to model formulation and data interpretation issues. In this paper, we present the weighted Weibull-G (WW-G) as a novel family of weighted distributions that may be used to solve problems in a variety of fields, including reliability, meta analysis, biomedicine, ecology, and others. Some statistical features that hold out of any baseline model are explained using general formulations. Four useful models are offered for the new family. Diverse density function shapes, such as symmetric, uni-modal, right skewed, U-shaped, or J-shaped, are represented, as well as different hazard rate shapes. The maximum likelihood estimators for family's parameters are derived. Monte Carlo simulations are used to examine the behavior of the estimators for one specific model, which is the WW-exponential. Finally, real data depicting the proportion of primary energy consumption produced from renewables in 75 country is used to demonstrate the flexibility of one model. Another real data analysis of global Carbon Dioxide (CO2) emissions per person in 2020 is taken into account in 211 country. The results of applications reveal that the weighted Weibull exponential distribution can, in reality, better match the data when compared to other competing distributions.

Keywords: Weighed Weibull distribution; Order statistics; Rényi entropy; Maximum likelihood Estimation.

1. Introduction

Ordinary distributions are not always suitable to model and forecast real data in many scenarios. This is especially true in the fields of engineering, economics, biology, and environmental science. As a result, several researchers have devised a number of extensions or generalizations to improve the desired aspects of probability distributions. As a result, several studies have proposed extensions or generalizations to improve the desirable aspects of probability distributions. Several authors proposed some of the generated families of continuous distributions, our interest here with the transformed-transformer (T-X) family prepared by Alzaatreh et al. (2013), based on T-X family several generated families have been regarded, see for instance Bourguignon et al. (2014), Hassan and Hemeda (2016), Hassan and Elgarhy (2016 a, b), Hassan et al. (2017 a, b), Cordeiro et al. (2016, 2017), Haq and Elgarhy (2018), and Hassan and Nassr (2018, 2019), among others.

Alzaatreh et al. (2013), proposed T-X family of distributions with the following cumulative distribution function (cdf)

$$F(x)=\int_0^{W(G(x))} r(t)\ dt, \quad (1)$$

where, $r(t)$ is the density function of a random variables T and $W(G(x))$ be a function of the cdf of any random variables X.

The theory of weighted distributions (WDs) offers a collective access for the problems of model specification and data interpretation challenges. When samples can be taken from both the original and developed distributions, it provides a technique for fitting models to unknown weight functions. By altering the probabilities of the actual occurrence of events to get at a specification of the probabilities of those events as observed and recorded, weighted distributions take into

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distributions.Since, a lot of essays on the issue have been published. Many authors have proposed and explored various WDs for diverse aims, for example; the reader can refer to Patil and Ord (1975, 1977); Rao, (1985); Ghitany et al. (2011); Kersey and Oluyede, (2012); Nasiru (2015), Ahmad et al. (2016); Sen et al. (2017); Abdul-Moniem and Diab (2018) and Hassan et al. (2021).

The Weibull distribution has gotten a lot of attention in the literature, due to its superiority over other distributions in simulating lifetime data. Nasiru (2015) introduced the weighted form of Weibull distribution with the following cdf

consideration the technique of ascertainment. For the improvement of accurate statistical models, weighted distributions appear frequently in research connected to reliability, analysis of family data, meta analysis, biomedicine, ecology, and other fields.

The concept of WDs was provided by Fisher (1934) and Rao (1965). Fisher (1934) studied how the methods of ascertainment can influence the form of the distribution of recorded observations, and Rao (1965) introduced and formulated it in general terms in connection with modelling statistical data when the usual practice of using standard distributions was found to be unsuitable. The probabilities of the events as observed and transcribed are modulated using weighted

$$G_{\mathrm{WW}}\left(t;\alpha,\theta,\lambda\right)=1-e^{-\left(\alpha t^{\theta}+\alpha\lambda^{\theta}t^{\theta}\right)} \quad t,\alpha,\lambda\ ,\ \theta>0. \tag{2}$$

The associated probability density function (pdf) of the WW distribution is as follows

$$g_{\mathrm{WW}}\left(t;\alpha,\theta,\lambda\right)=\left(1+\lambda^{\theta}\right)\alpha\theta t^{\theta-1}e^{-\left(\alpha t^{\theta}+\alpha\lambda^{\theta}t^{\theta}\right)},\quad t,\alpha,\lambda\ ,\ \theta>0. \tag{3}$$

'substitution method' which attempts to correct for the inefficiencies in fossil fuel production. We consider another practical data analysis representing the global CO2 emissions per person in 2020 in 211 country. Emissions of CO2 are from burning oil, coal, and gas for energy use, burning wood and waste materials, and from industrial processes such as cement production. This is leading to an increase in the earth's surface temperature and the related effects on the climate, sea level rise, and world agriculture. It has been demonstrated that the special model of this family produces useful results when compared to other models.

Let $G(x;\xi)$, and $g(x;\xi)$ be the baseline cdf and pdf, respectively, for a random variable X. The cdf of the WW-G family are given by utilizing the T-X generator defined in (1) and using the WW distribution for $\alpha=1$ as below

Nasiru (2015) examined a number of characteristics of the WW distribution and calculated model parameters. Furthermore, the WW distribution's utility was proved by applying it to a real-life dataset.

The paper's main goal is to propose a new family of distributions based on the WW distribution.The WW-
G family is the name given to the new family and a comprehensive description of its mathematical features is given. In fact, the efficiency to model data with increasing, decreasing, unimodal, and U-shaped, or J-shaped failure rates is the driving force behind the WW-G family. Furthermore, the versatility of one particular model is demonstrated through practical data analysis related to statistics in 2019, the share of primary energy consumption that comes from renewable technologies in 75 country. Note that this data is based on primary energy calculated by the

$$F_{\mathrm{WW}\text{-}G}\left(x;\lambda,\theta,\zeta\right)=\theta\left(1+\lambda^{\theta}\right)\int_{0}^{G(x;\zeta)/\bar{G}(x;\zeta)}t^{\theta-1}\mathrm{e}^{-\left(1+\lambda^{\theta}\right)t^{\theta}}\,dt=1-\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left(G(x;\zeta)/\bar{G}(x;\zeta)\right)^{\theta}},$$

(4) where, $\lambda,\theta,$ are scale and shape parameters respectively. Therefore, the pdf of the WW-G family is given by

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$$f_{\text{WW-}G}\left(x;\lambda,\theta,\zeta\right)=\theta\left(1+\lambda^{\theta}\right)g\left(x;\zeta\right)G^{\theta-1}\left(x;\zeta\right)\left(\bar{G}\left(x;\zeta\right)\right)^{-(\theta+1)}\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left(G(x;\zeta)/\bar{G}(x;\zeta)\right)^{\theta}}.$$
(5)

Hereafter, a random variable X has pdf (5), is denoted, by $X\sim$ WW-G.

The reliability function and hazard rate function, are respectively, given by

- As a novel family, the weighted Rayleigh-G (WR-G) is obtained for $\theta=2$.
- As a result, we get a new family called the weighted exponential-G (WE-G) for $\theta=1$.

$$R_{\text{WW-}G}\left(x;\lambda,\theta,\zeta\right)=\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left(G(x;\zeta)/\bar{G}(x;\zeta)\right)^{\theta}},$$

and

$$h_{\text{WW-}G}\left(x;\lambda,\theta,\zeta\right)=\theta\left(1+\lambda^{\theta}\right)g\left(x;\zeta\right)G^{\theta-1}\left(x;\zeta\right)\left(1-G\left(x;\zeta\right)\right)^{-(\theta+1)}.$$

The following are the possible classifications for this publication. Section 2 derives some of the family's general statistical features. The maximum likelihood (ML) technique is used in Section 3 to estimate the parameters of the family. Section 4 looks at four new WW-G sub-models. In Section 5, a simulation study is conducted to estimate model parameters for one distribution. Section 6 investigates an illustration aim using real data. Finally, some closing notes are made throughout the article

2. The WW-G Properties

In this part, we derive ordinary and incomplete moments, quantile function, entropy measure, order statistics, and probability weighted moments (PWMs), among other properties of the WW-G family.

2.1 Quantile Function

Let X denotes a random variable has the pdf (5), the quantile function; say $Q(u)$ of X is given by:

$$Q_{\text{WW-}G}\left(u\right)=G^{-1}\left[\left(\frac{1}{1+\lambda^{\theta}}\ln\left(\frac{1}{1-u}\right)\right)^{\frac{1}{\theta}}\Bigg/1+\left(\frac{1}{1+\lambda^{\theta}}\ln\left(\frac{1}{1-u}\right)\right)^{\frac{1}{\theta}}\right],$$

where, u is a uniform distribution on the interval (0,1) and G^{-1} is the inverse function of $G(.)$, also, we get the median by inserting u = 0.5.

2.2 A Valuable Representation

The pdf and cdf representations for the WW-G distribution are shown here. Since, the exponential expansion has been written as this:

$$e^{-ax}=\sum_{i=0}^{\infty}\frac{(-1)^{i}a^{i}}{i!}x^{i}. \qquad (6)$$

The pdf of the WW-G distribution is then obtained by using the exponential expansion (6) in (5)

$$f_{\text{WW-}G}\left(x;\Phi\right)=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\theta\left(1+\lambda^{\theta}\right)^{i+1}g\left(x;\zeta\right)\left(G\left(x;\zeta\right)\right)^{\theta(i+1)-1}\left(1-G\left(x;\zeta\right)\right)^{-(\theta(i+1)+1)},$$
(7)

where, $\Phi=(\lambda,\theta,\zeta)$. Since the generalized binomial theorem is already known,

$$\left(1-z\right)^{-\beta}=\sum_{j=0}^{\infty}\binom{\beta+j-1}{j}z^{j}. \qquad (8)$$

Hence, by using(8) in (7), the WW-G pdf can be written as follows

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$$f_{\text{WW-}G}\left(x;\Phi\right)=\sum_{i,j=0}^{\infty} w_{i,j}\, g\left(x;\zeta\right)\left(G\left(x;\zeta\right)\right)^{j+(i+1)\theta-1}, \tag{9}$$

where,

$$w_{i,j}=\frac{(-1)^{i}}{i!}\theta\left(1+\lambda^{\theta}\right)^{i+1}\binom{\theta(i+1)+j}{j}.$$

Further, an expansion for $\left[F_{\text{WW-}G}\left(x;\Phi\right)\right]^{h}$ is derived, for h is integer, again, the exponential expansion and the binomial expansion is worked out.

$$\left[F_{\text{WW-}G}\left(x;\Phi\right)\right]^{h}=\sum_{z=0}^{\infty} S_{z} G\left(x;\zeta\right)^{m\theta+z}, \tag{10}$$

where,

$$S_{z}=\sum_{k=0}^{h}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}k^{m}\left(1+\lambda^{\theta}\right)^{m}\binom{h}{k}\binom{m\theta+z-1}{z}.$$

2.3 Probability Weighted Moments

The PWM is another method for obtaining the moments of statistical distributions whose inverse form cannot be represented simply. The PWM, denoted by $\tau_{r,s}$, is calculated for a random variable X using pdf (9) and cdf (10)

$$\tau_{r,s}=E[X^{r}F(x)^{s}]=\int_{-\infty}^{\infty} x^{r} f(x)(F(x))^{s}\,dx. \tag{11}$$

The PWM of WW-G is produced by replacing h with s and substituting(9) and (10) in (11)

$$\tau_{r,s}=\int_{-\infty}^{\infty}\sum_{i,j,z=0}^{\infty} s_{z} w_{i,j} x^{r} g(x;\zeta)(\mathrm{G}(x;\zeta))^{z+j+(m+i+1)\theta-1}dx.$$

Then,

$$\tau_{r,s}=\sum_{i,j,z=0}^{\infty} s_{z} w_{i,j}\tau_{r,z+j+(m+i+1)\theta-1}.$$

2.4 Moments

In general, we must always remember the importance of moments in any statistical study, especially in applied fields. Moments, for example, can be used to investigate important aspects of a distribution such as tendency, dispersion, skewness, and kurtosis. If X has the pdf (9), we can calculate the r^{th} moment as follows:

$$\mu_{r}'=E[X^{r}]=\int_{-\infty}^{\infty}\sum_{i,j=0}^{\infty} w_{i,j} x^{r} g(x;\zeta)(G(x;\zeta))^{j+(i+1)\theta-1}dx.$$

Then,

$$\mu_{r}'=\sum_{i,j=0}^{\infty} w_{i,j}\tau_{r,j+(i+1)\theta-1}.$$

For a random variable X it is known that, the moment generating function is defined as

$$M_{X}(t)=\sum_{r=0}^{\infty}\frac{t^{r}}{r!}\mu_{r}'=\sum_{r,i,j=0}^{\infty}\frac{t^{r}}{r!}w_{i,j}\tau_{r,j+(i+1)\theta-1}.$$

2.5 Order Statistics

Many fields of statistics, including as reliability and life testing, have extensively used order statistics. Let X_1, $X_2,\ldots,X_n$ with their corresponding continuous distribution function be independent and identically distributed (i.i.d) random variables. Allow an ordered random sample $X_{(1)}$, $X_{(2)},\ldots,X_{(n)}$ from a population of size n to be generated. The s^{th} order statistic's pdf is defined as follows:

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$$f_{s:n}(x)=D(n,s)\,f(x)\sum_{v=0}^{n-s}(-1)^{v}\binom{n-s}{v}F(x)^{v+s-1}, \tag{12}$$

where, $D(n,s)=n!/(n-s)!(s-1)!$. The pdf of the s^{th} order statistic for the WW-G family is obtained by replacing h with $v+s-1$ and substituting (9) and(10) in (12).

$$f_{s:n}(x)=\frac{g(x;\zeta)}{B(s,n-s+1)}\sum_{v=0}^{n-s}\sum_{i,j,z=0}^{\infty}p_{z,v}\,\mathrm{G}(x;\zeta)^{z+j+(m+i+1)\theta-1}, \tag{13}$$

where $p_{z,v}=(-1)^{v}\binom{n-s}{v}w_{i,j}s_{z}$, $g(.)$ and $G(.)$ are the pdf and cdf of any baseline distribution, respectively.

Further, the r^{th} moment of s^{th} order statistics for WW-G family is defined by:

$$E(X_{s:n}{}^{r})=\frac{1}{B(s,n-s+1)}\sum_{v=0}^{n-s}\sum_{i,j,z=0}^{\infty}p_{z,v}\int_{-\infty}^{\infty}x^{r}g(x;\zeta)\,\mathrm{G}(x;\zeta)^{z+j+(m+i+1)\theta-1}dx.$$

Then,

$$E(X_{s:n}{}^{r})=\frac{1}{B(s,n-s+1)}\sum_{v=0}^{n-s}\sum_{i,j,z=0}^{\infty}p_{z,v}\tau_{z+j+(m+i+1)\theta-1}.$$

3. Parameter Estimation

On the basis of complete samples, this section discusses ML estimators of unknown parameters for the WW-G family of distributions. Let X_1, X_2,..., X_n represent the observed values from the WW-G family with a given set of parameters $\Phi=(\lambda,\theta,\zeta)^{T}$. The log-likelihood function for parameter vector is obtained as follows

$$\ln L(\Phi)=n\ln\theta+n\ln\left(1+\lambda^{\theta}\right)+\sum_{i=1}^{n}\ln g\left(x_i;\zeta\right)+(\theta-1)\sum_{i=1}^{n}\ln G\left(x_i;\zeta\right)$$
$$-(\theta+1)\sum_{i=1}^{n}\ln\left(1-G\left(x_i;\zeta\right)\right)-\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}\left(G\left(x_i;\zeta\right)/\bar{G}\left(x;\zeta\right)\right)^{\theta}.$$

The elements of the score function $U(\Phi)=(U_{\lambda},U_{\theta},U_{\zeta_k})$ are given by

$$U_{\lambda}=\frac{n\theta\lambda^{\theta-1}}{1+\lambda^{\theta}}-\theta\lambda^{\theta-1}\sum_{i=1}^{n}\left(G\left(x_i;\zeta\right)/\bar{G}\left(x;\zeta\right)\right)^{\theta},$$

$$U_{\theta}=\frac{n}{\theta}+\frac{n\lambda^{\theta}\ln\lambda}{1+\lambda^{\theta}}+\sum_{i=1}^{n}\ln G\left(x_i;\zeta\right)-\sum_{i=1}^{n}\ln\left(1-G\left(x_i;\zeta\right)\right)-\lambda^{\theta}\ln\lambda\sum_{i=1}^{n}\left(G\left(x_i;\zeta\right)/\bar{G}\left(x;\zeta\right)\right)^{\theta}$$
$$-\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}\left(G\left(x_i;\zeta\right)/\bar{G}\left(x;\zeta\right)\right)^{\theta}\ln\left(G\left(x_i;\zeta\right)/\bar{G}\left(x;\zeta\right)\right),$$

and,

$$U_{\zeta_k}=\sum_{i=1}^{n}\frac{\partial/\partial_{\zeta_k}g\left(x_i;\zeta\right)}{g\left(x_i;\zeta\right)}+(\theta-1)\sum_{i=1}^{n}\frac{\partial/\partial_{\zeta_k}G\left(x_i;\zeta\right)}{G\left(x_i;\zeta\right)}+(\theta+1)\sum_{i=1}^{n}\frac{\partial/\partial_{\zeta_k}G\left(x_i;\zeta\right)}{1-G\left(x_i;\zeta\right)}$$
$$-\theta\left(1+\lambda^{\theta}\right)\sum_{i=1}^{n}\left(\frac{\partial/\partial_{\zeta_k}G\left(x_i;\zeta\right)\left(G\left(x_i;\zeta\right)\right)^{\theta-1}}{\left(1-G\left(x_i;\zeta\right)\right)^{\theta+1}}\right).$$

The ML estimator of is obtained by setting U_{λ},U_{θ} and U_{ζ_k} equal to zeros and solving these equations at the same time. These equations cannot be solved analytically, however they can be solved numerically using iterative methods employing statistical software.

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uniform, WW-exponential, WW-Rayleigh and WW-Kumaraswamy.

4. Sub-Models

In this section, we define new four sub-models of the WW-G family namely, WW-

4.1 WW-uniform distribution

The cdf of WW-uniform (WWU) is derived from (4), by taking $g(x;\beta)=\beta^{-1},\ \ 0<x<\beta$ and $G(x;\beta)=x\beta^{-1}$ as the following

$$F_{\mathrm{WWU}}(x;\lambda,\theta,\beta)=1-\mathrm{e}^{-\left(1+\lambda^{\theta}\right)x^{\theta}(\beta-x)^{-\theta}}.$$

The corresponding pdf is given by

$$f_{\mathrm{WWU}}(x;\lambda,\theta,\beta)=\beta\theta\left(1+\lambda^{\theta}\right)x^{\theta}(\beta-x)^{-(\theta+1)}\,\mathrm{e}^{-\left(1+\lambda^{\theta}\right)x^{\theta}(\beta-x)^{-\theta}}.\ 0<x<\beta.$$

Moreover, the hazard rate functions is given by,

$$h_{\mathrm{WWU}}(x;\lambda,\theta,\beta)=\beta\theta\left(1+\lambda^{\theta}\right)x^{\theta}(\beta-x)^{-(\theta+1)}.$$

4.2 WW-Exponential Distribution

The cdf and pdf of WW-exponential (WWE) distribution are derived from (4) and (5) taking $G(x;\alpha)=1-e^{-\alpha x}$ as the following

$$F_{\mathrm{WWE}}(x;\lambda,\theta,\alpha)=1-\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left(e^{\alpha x}-1\right)^{\theta}}.$$

and,

$$f_{\mathrm{WWE}}(x;\lambda,\theta,\alpha)=\theta\alpha\left(1+\lambda^{\theta}\right)e^{\alpha x}\left(e^{\alpha x}-1\right)^{\theta-1}\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left(e^{\alpha x}-1\right)^{\theta}}.\ x>0\ ,\ \alpha,\lambda,\theta>0.$$

Further, the hazard rate function is as follows

$$h_{\mathrm{WWE}}(x;\lambda,\theta,\alpha)=\theta\alpha\left(1+\lambda^{\theta}\right)e^{\alpha x}\left(e^{\alpha x}-1\right)^{\theta-1}.$$

4.3 WW –Rayleigh Distribution

The cdf and pdf of WW-Rayleigh (WWR) distribution are derived from (4) and (5) taking $G(x;\alpha)=1-e^{-\alpha x^{2}}$ as the following

$$F_{\mathrm{WWR}}(x;\lambda,\theta,\alpha)=1-\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left(e^{\alpha x^{2}}-1\right)^{\theta}}.$$

and,

$$f_{\mathrm{WWR}}(x;\lambda,\theta,\alpha)=2\theta\alpha x\left(1+\lambda^{\theta}\right)e^{\alpha x^{2}}\left(e^{\alpha x^{2}}-1\right)^{\theta-1}\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left(e^{\alpha x^{2}}-1\right)^{\theta}}.\ x>0\ ,\ \alpha,\lambda,\theta>0.$$

Further, the hazard rate function is as follows

$$h_{\mathrm{WWR}}(x;\lambda,\theta,\alpha)=2\theta\alpha x\left(1+\lambda^{\theta}\right)e^{\alpha x^{2}}\left(e^{\alpha x^{2}}-1\right)^{\theta-1}.$$

4.4 WW - Kumaraswamy distribution

The Kumaraswamy distribution has been suggested by (Kumaraswamy, 1980). The cdf and pdf of Kumaraswamy distribution are given by

$$G(x;a,b)=1-\left(1-x^{a}\right)^{b}\quad,\ g(x;a,b)=abx^{a-1}\left(1-x^{a}\right)^{b-1},0<x<1,a,b>0$$

The cdf, pdf, and the hazard rate functions for WW- Kumaraswamy distribution (WWKw) are obtained from (4) and (5), respectively as

$$F_{\mathrm{WWKw}}(x;a,b,\lambda,\theta)=1-\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left((1-x^{a})^{-b}-1\right)^{\theta}},0<x<1\ ,\ a,b,\lambda,\theta>0.$$

$$f_{\mathrm{WWKw}}(x;a,b,\lambda,\theta)=ab\theta\left(1+\lambda^{\theta}\right)x^{a-1}(1-x^{a})^{-b-1}\left((1-x^{a})^{-b}-1\right)^{\theta-1}\mathrm{e}^{-\left(1+\lambda^{\theta}\right)\left((1-x^{a})^{-b}-1\right)^{\theta}},$$

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and,

$$h_{\mathrm{WWKw}}\left(x;a,b,\lambda,\theta\right)=ab\theta\left(1+\lambda^{\theta}\right)x^{a-1}(1-x^{a})^{-b-1}\left((1-x^{a})^{-b}-1\right)^{\theta-1}.$$

The plots of pdf and hazard rate function for the (a) WWU, (b) WWE, (c) WWR and (d) WWKw distributions are given in Figures 1 and 2 respectively

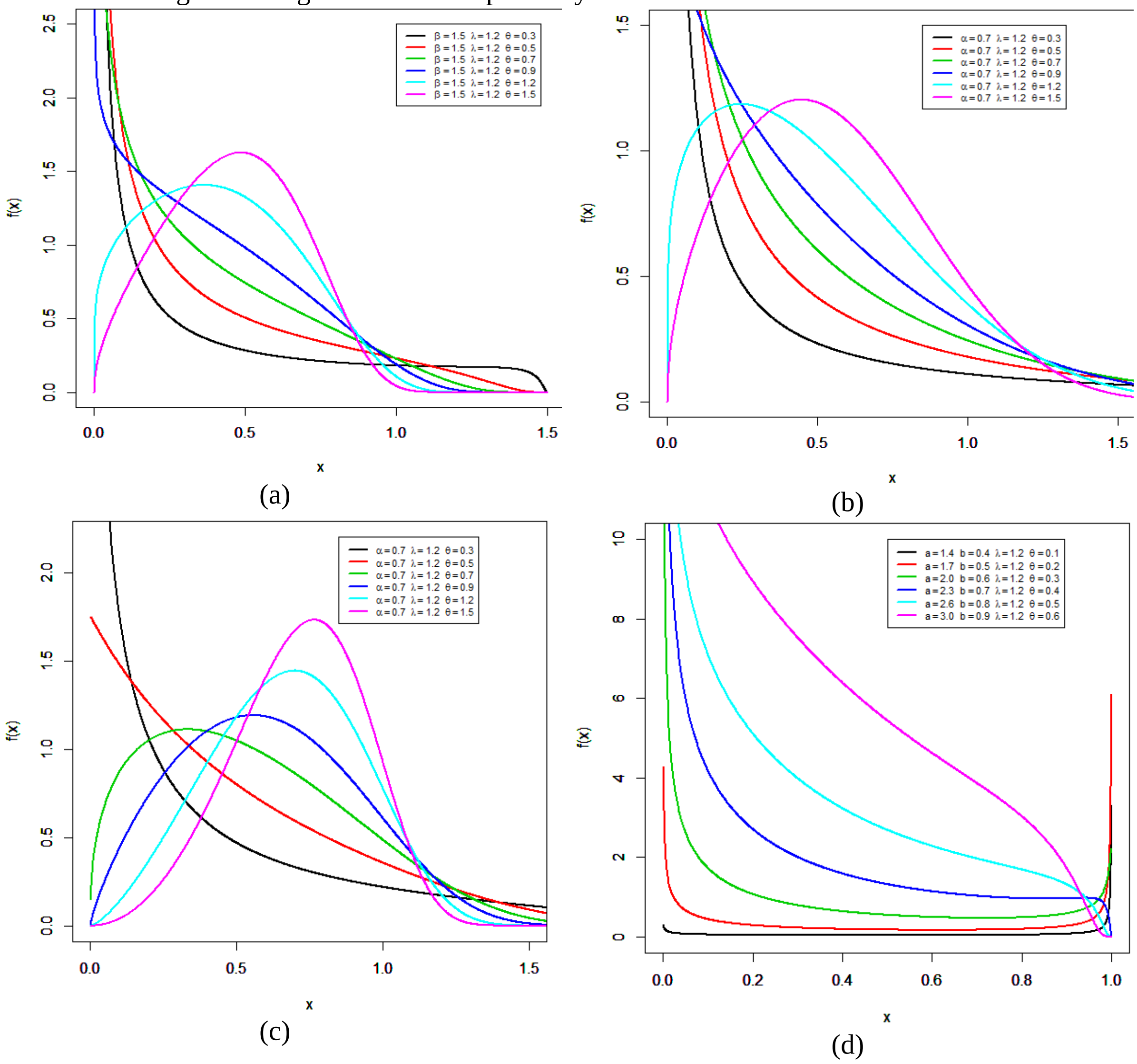
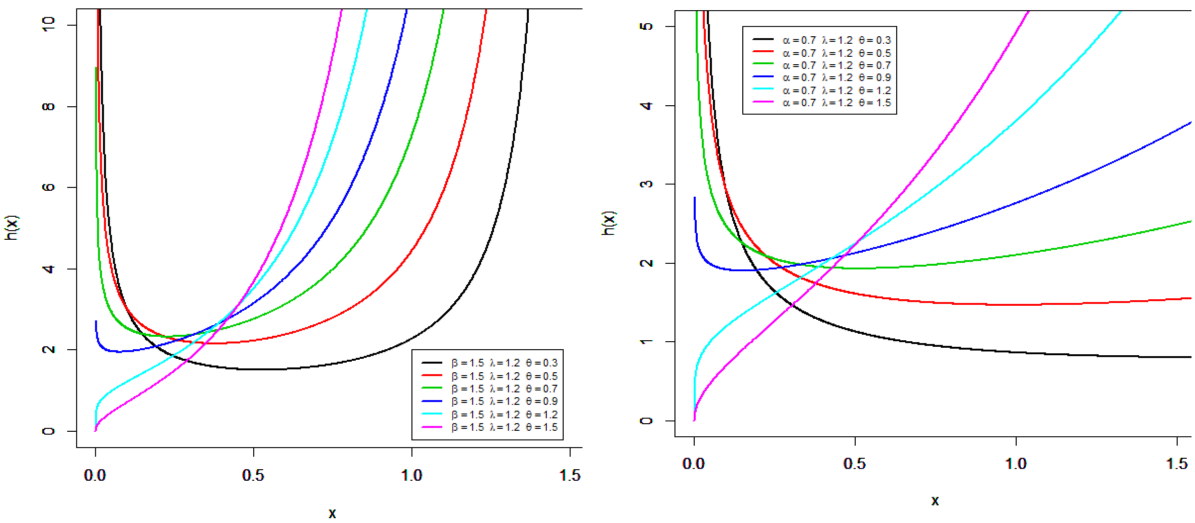


Figure 1: The pdfs of (a) WWU, (b)WWE, (c) WWR and (d) WWKw distributions



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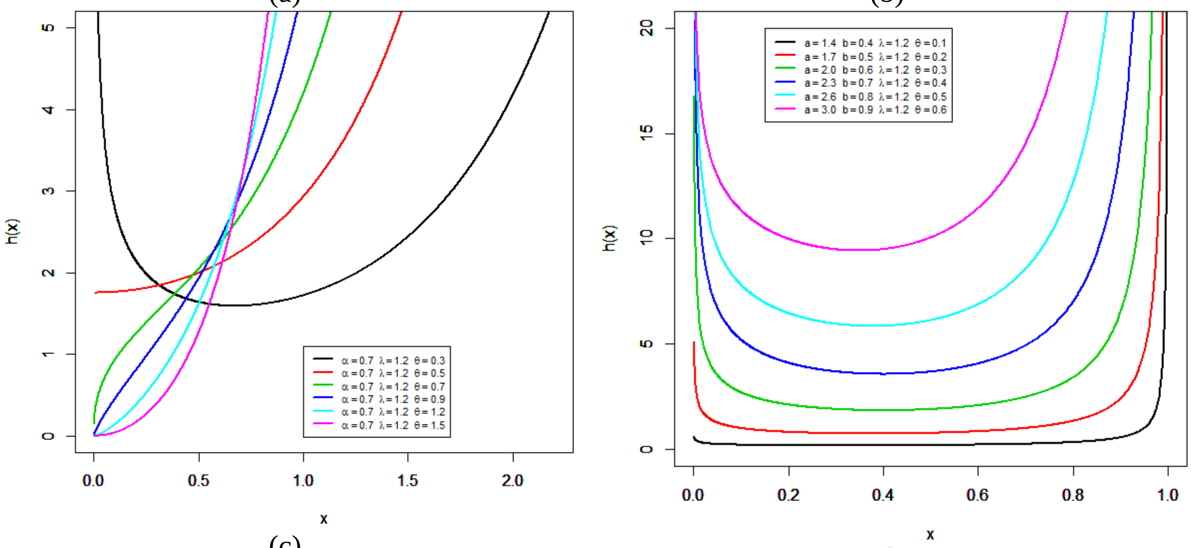
(a) (b)



(c) (d)

Figure 2: The hrfs of (a) WWU, (b)WWE, (c) WWR and (d) WWKw distributions

- Set1(α = 0.5, λ= 0.5, θ = 0.5), Set2(α =0.75, λ=0.5 ,θ =1.0) ,Set3(α =1.25, λ=0.5 ,θ =1.5) ,Set4(α =0.5, λ=0.75 ,θ =1.5), Set5(α =0.5, λ=1.25 ,θ =1.75), and Set6(α =0.5, λ=1.5 ,θ =0.8) are the six sets of parameters that are considered.
- For each parameter value and sample size, the WWE model's MLEs are evaluated.
- Obtain the means, biases, and MSEs of the MLE for different sets of parameters and for each sample size by repeating this process 10,000 times.
- Tables 1 to 3 show the empirical results. These tables show that as sample sizes grow, the estimates remain rather steady and close to the true value of the parameters.

We can deduce from Figure 1 that the pdf of the distributions can have right and left skewness, bathtub, uni-model, and symmetric properties. As shown in Figure 2, the hrf of distributions can be U-shaped, J-shaped, reversed J-shaped.

5. Simulation Study

Herein, we evaluate the ML estimates' performance in terms of sample size *n*. The performance of ML estimates (MLEs) for the WWE model, which is one of the family's sub-models, is evaluated numerically. The Mathematic programme is used to evaluate estimates based on the following quantities for each sample size: biases and empirical mean squared errors (MSEs). The following are the numerical procedures:

- A random sample X_1, X_2,..., X_n is evaluated, with n = 30, 50, 100, and 150. These random samples are generated using the inversion method from the WWE distribution.

Table 1: MLE, Bias and MSE of WWE model for Set1and Set2

n	Paramete r	Set1(α = 0.5, λ= 0.5, θ = 0.5)			Set2(α =0.75, λ=0.5 ,θ =1.0)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	α	0.5251	0.0251	0.0129	0.9584	0.2084	0.3067
	λ	0.5013	0.0013	0.0010	0.5073	0.0073	0.0028
	θ	0.6168	0.1168	0.4761	1.2009	0.2008	0.6600
50	α	0.5106	0.0106	0.0055	0.7987	0.0487	0.0499
	λ	0.5004	0.0004	0.0004	0.5008	0.0008	0.0009
	θ	0.5287	0.0287	0.0198	1.0729	0.0729	0.0846

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100	α	0.5065	0.0065	0.0033	0.7772	0.0272	0.0284
	λ	0.5007	0.0007	0.0002	0.5003	0.0003	0.0006
	θ	0.5141	0.0141	0.0101	1.0304	0.0304	0.0425
150	α	0.5049	0.0049	0.0016	0.7697	0.0197	0.0134
	λ	0.5005	0.0005	0.0001	0.5010	0.0010	0.0003
	θ	0.5096	0.0096	0.0046	1.0196	0.0196	0.0196

Table 2: MLE, Bias and MSE of WWE model for Set3and Set4

n	Parameter	Set3(α =1.25, λ=0.5 ,θ =1.5)			Set4(α =0.5, λ=0.75 ,θ =1.5)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	α	1.5182	0.2682	1.0400	0.6128	0.1128	0.3176
	λ	0.5025	0.0025	0.0013	0.7588	0.0088	0.0062
	θ	1.5834	0.0834	0.1657	1.6007	0.1007	0.1777
50	α	1.4043	0.1543	0.4244	0.5667	0.0667	0.0635
	λ	0.5017	0.0017	0.0008	0.7576	0.0076	0.0043
	θ	1.5538	0.0538	0.0924	1.5460	0.0460	0.0956
100	α	1.3096	0.0596	0.0955	0.5204	0.0204	0.0152
	λ	0.5006	0.0006	0.0004	0.7509	0.0009	0.0021
	θ	1.5195	0.0195	0.0420	1.5257	0.0257	0.0447
150	α	1.2838	0.0338	0.0548	0.5182	0.0182	0.0092
	λ	0.5001	0.0001	0.0002	0.7527	0.0027	0.0013
	θ	1.52248	0.02248	0.02756	1.5149	0.0149	0.0266

Table 3: MLE, Bias and MSE of WWE model for Set5and Set6

n	Parameter	Set5(α =0.5, λ=1.25 ,θ =1.75)			Set6(α =0.5, λ=1.5 ,θ =0.8)		
		MLE	Bias	MSE	MLE	Bias	MSE
30	α	0.6439	0.1439	1.0302	0.5190	0.0190	0.0100
	λ	1.2933	0.0433	0.0693	1.5106	0.0106	0.0159
	θ	1.8412	0.0912	0.2545	0.8577	0.0577	0.0544
50	α	0.5829	0.0829	0.0922	0.5115	0.0115	0.0064
	λ	1.2856	0.0356	0.0397	1.5071	0.0070	0.0103
	θ	1.8103	0.0603	0.1307	0.8250	0.0250	0.0290
100	α	0.5325	0.0325	0.0221	0.5061	0.0061	0.0031
	λ	1.2649	0.0149	0.0189	1.5037	0.0037	0.0052
	θ	1.7770	0.0270	0.0557	0.8140	0.0140	0.0127
150	α	0.5193	0.0193	0.0117	0.5041	0.0041	0.0021
	λ	1.2594	0.0094	0.0117	1.5029	0.0029	0.0034
	θ	1.7601	0.0101	0.0338	0.8085	0.0085	0.0077

weighted Weibull (WW), length biased Weighted Weibull (LBWW), Exponentiated Exponential Weibull (EEW), and the Exponential (E). We obtain the MLEs, and standard errors (SEs) of the model parameters. To compare the distribution models, we consider criteria like; Akaike information

6. A Technical analysis reliable renewable energy sources to reduce co2 emissions

In this section, two data sets are studied to show how the WWE distribution outperforms other models. Comparing the new model to some models; namely, Gamma-Weibull (GW),

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The second data represents the global CO2 emissions per person in 2020 in 211 country, as mentioned in [https://www.statista.com/statistics/270508/co2-emissions-per-capita-by-country/]. Since the industrial revolution, the burning of carbon-based fuels has dramatically increased atmospheric carbon dioxide concentrations, hastening global warming and generating anthropogenic climate change. Because it dissolves in water to generate carbonic acid, it is a major contributor to ocean acidification. The addition of man-made greenhouse gases to the atmosphere causes the earth's radioactive balance to be disrupted. As a result, the earth's surface temperature is rising, affecting the climate, sea level rise, and global agriculture. CO2 emissions are produced by burning fossil fuels such as oil, coal, and gas for energy, as well as burning wood and waste materials and industrial operations such as cement manufacturing.

Some descriptive statistics of the both data sets are mentioned in Table 4.

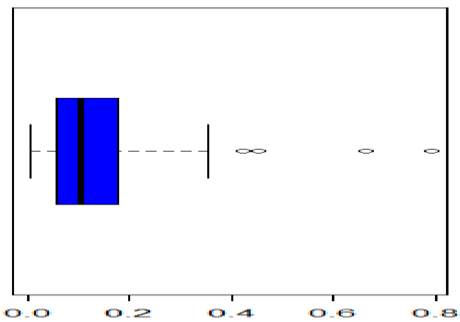
criterion (AIC), the consistent AIC (CAIC), Bayesian IC (BIC), Hannan-Quinn IC (HQIC), Kolmogorov–Smirnov (KS) test and p-value (PV) test. The wider distribution, on the other hand, refers to lower AIC, CAIC, BIC, HQIC, KS, and the greatest value of PV.

In the first data, we looked at statistics from 75 countries in 2019 to see what percentage of primary energy consumption came from renewable technologies—a mix of hydropower, solar, wind, geothermal, wave, tidal, and modern biofuels [traditional biomass, which can be a significant energy source in low-income countries, is not included]. This data is based on primary energy estimated using the "substitution approach," which attempts to account for inefficiencies in fossil fuel production. As stated in [https://ourworldindata.org/renewable-energy], it does this by transforming non-fossil fuel sources into their "input equivalents," or the amount of primary energy that would be required to produce the same quantity of energy if it came from fossil fuels.

Table 4: Descriptive statistics for the both data sets

	n	Mean	Median	Mode	Variance	Skewness	Kurtosis	Minimum	Maximum
Data I	75	0.14	0.1054	0.0618	0.02	2.2642	6.8757	0.0027	0.7908
Data II	211	4.58	2.77	0.26	31.87	2.6127	8.8032	0.03	37.02

Figure 3 illustrates the boxplots for the proposed data. The total time test (TTT) plot (see Aarset (1987)) is an essential graphical technique to check if the data can be applied to a given distribution or not, this is the TTT plot's empirically determined version is given by plotting $T\left(\frac{r}{n}\right) = \sum_{i=1}^{n}[y_{i:n} + (n-r)y_{r:n}] / \sum_{i=1}^{n}(y_{i:n})$ against r/n, where $r = 1, \ldots, n$ and$y_{i:n}$, $(i = 1, \ldots, n)$ are the order statistics of the sample. The hrf is constant if the TTT plot is graphically displayed as a straight diagonal, but increasing (or decreasing) if the TTT plot is concave (or convex), according to Aarset (1987). If the TTT plot is initially convex and then concave, the hrf is U-shaped (bathtub); otherwise, the hrf is unimodal. Figure 4 shows the TTT plots for both data sets.The plots of the profile log likelihood for the both data sets are illustrated in Figures 5 and 6.



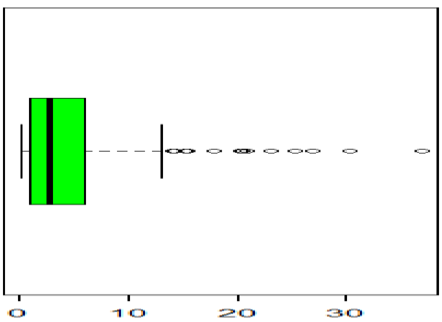


Figure 3: Boxplots with color blue for the first data and the color green for the second data

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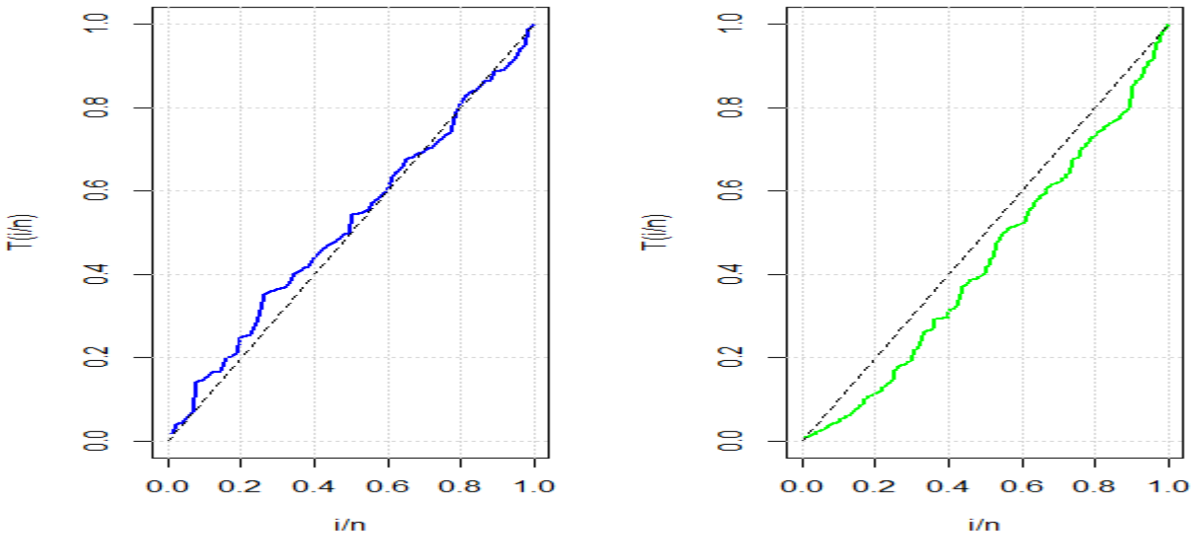

Figure 4: TTT plots with color blue for the first data and the color green for the second data

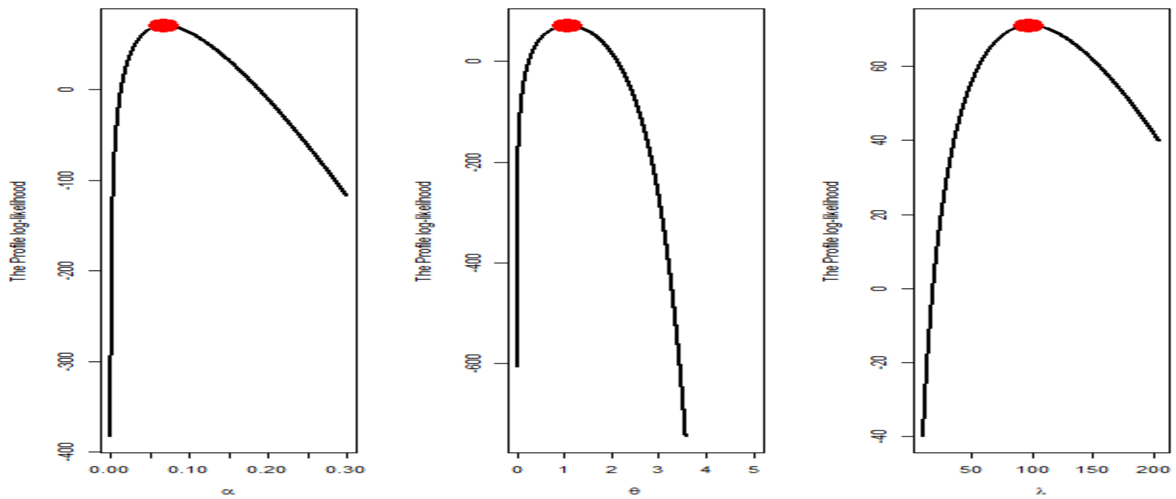

Figure 5. The profile log likelihood for the First data set

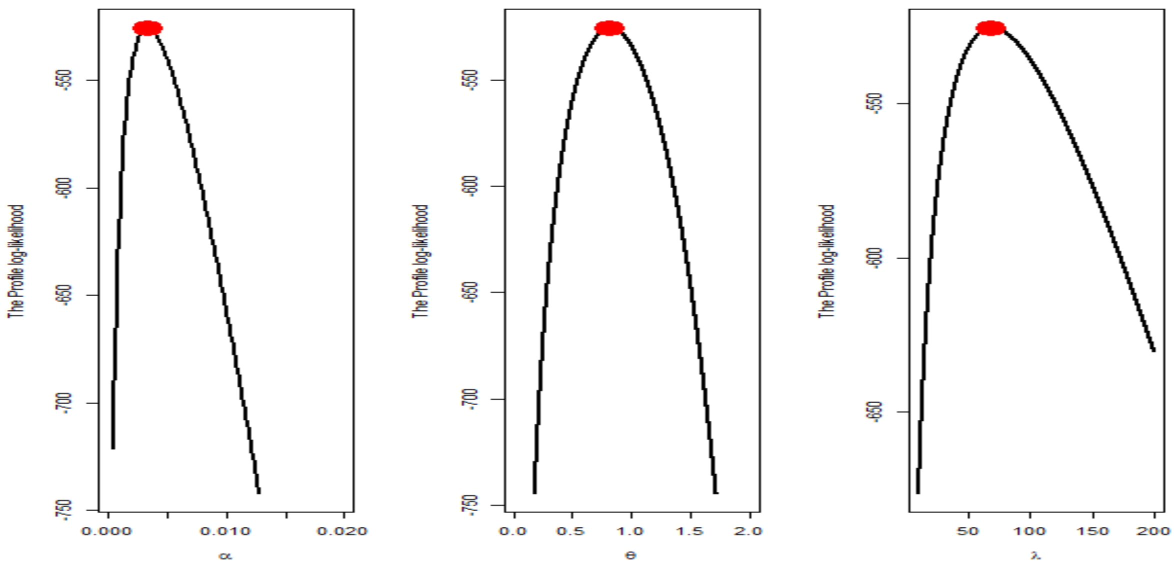

Figure 6. The profile log likelihood for the second data set

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The MLEs of the six competing models, as well as their SEs and AIC, CAIC, BIC, HQIC, PV, and KS values for both data sets, are shown in Tables 5 and 6.

Table 5. MLEs, SEs and measures of fitting for the first data set

Model	MLEs and S. E				AIC	CAIC	BIC	HQIC	KS	P-V
	α	θ	λ	β						
WWE	0.068	1.079	97.871		-136.143	-135.805	-136.517	-133.367	0.05405	0.98081
	0.201	0.097	293.335							
EEW	1.68	0.832	2.257	0.245	-135.221	-134.65	-135.721	131.52	0.0555	0.97498
	0.951	0.245	0.000019	0.000242						
WW	96.927	74.274	0.107		-131.989	-131.651	-132.364	-129.213	0.0908	0.56651
	103.818	92.257	0.067							
LBWW	0.057	0.436	0.759		-130.061	-129.723	-130.436	-127.285	0.08526	0.64683
	0.083	0.862	0.325							
GW	8.851	0.669	0.759		-135.248	-134.91	-135.623	-132.472	0.05622	0.97171
	2.168	0.789	0.325							
E	6.977				-132.558	-132.503	-132.683	-131.632	0.08272	0.68386
	0.806									

Table 6. The MLEs, SEs and measures of fitting for the second data set

Model	MLEs and S. E				AIC	CAIC	BIC	HQIC	KS	P-V
	α	θ	λ	β						
WWE	0.003307	0.824	69.429		1057	1058	1059	1062	0.03974	0.89282
	0.007718	0.048	169.258							
EEW	1.185	0.757	0.2	0.41	1061	1061	1063	1068	0.04561	0.77235
	0.402	0.147	4146	1122						
WW	85.315	89.506	0.073		1075	1075	1076	1079	0.09172	0.05745
	68.474	70.33	0.029							
LBWW	0.006964	1.711	0.3		1064	1064	1065	1068	0.07727	0.16088
	0.024	2.181	0.082							
GW	17.905	3.046	0.156		1070	1070	1071	1074	0.08698	0.08211
	11.438	0.937	0.047							
E	0.218				1066	1066	1067	1068	0.09273	0.05312
	0.015									

Moreover, the plots of empirical cdf and empirical pdf displayed in Figures 7 and 9 respectively. Furthermore, the PP plots of all competitive models for both data sets are displayed in Figures 6 and 8.

We find that the WWE distribution with three-parameter provides a better fit than the others five models. It has the smallest values of AIC, CAIC, BIC, HQIC, KS and the greatest value of PV among those considered here.

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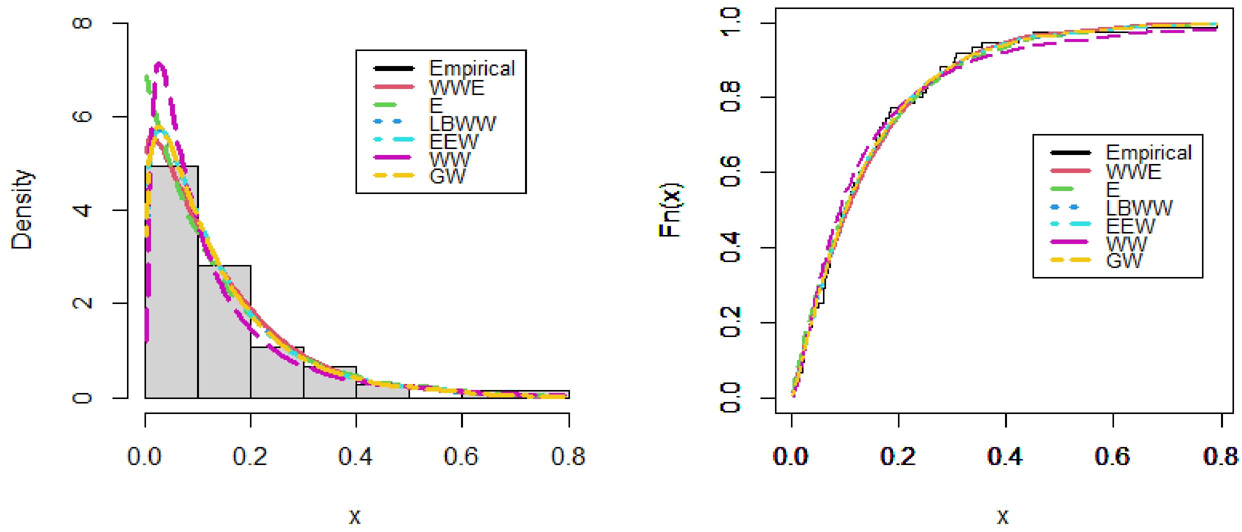

Figure 7. Estimated pdf and cdf of competitive models for the first dataset

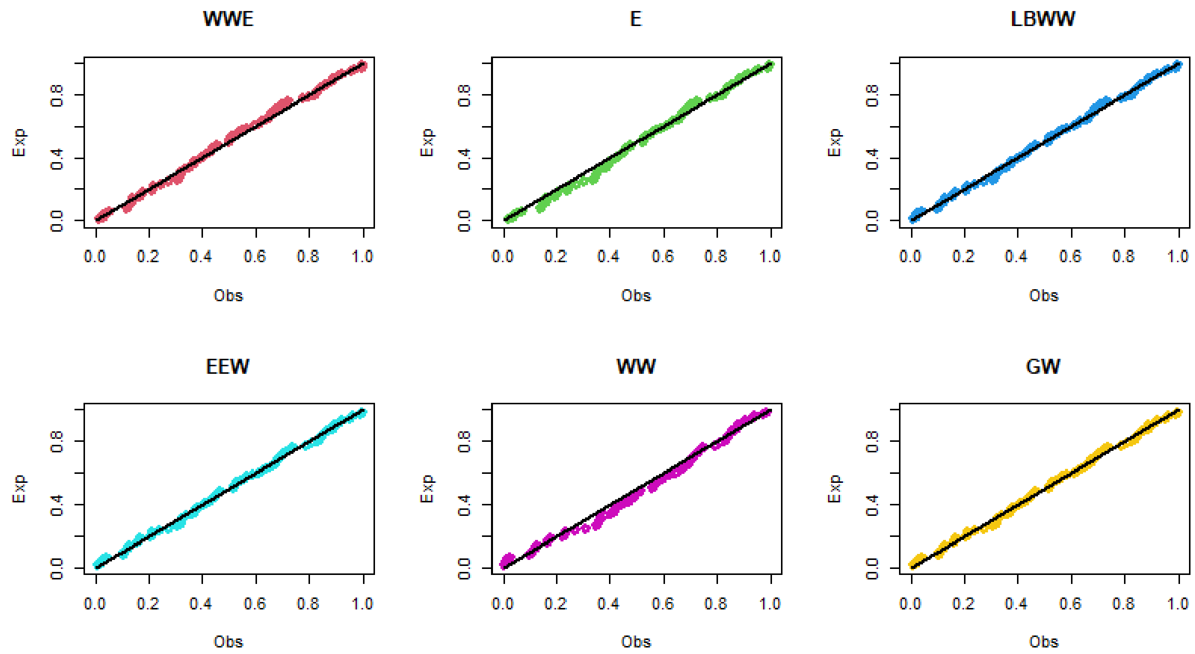

Figure 8. PP plots of the fitted models for the first data set

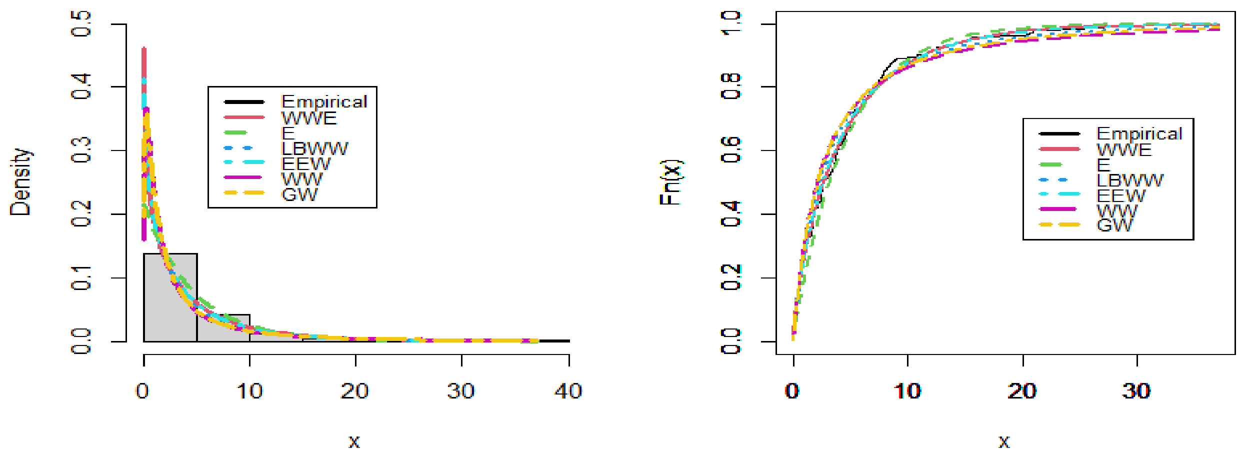

Figure 9. Estimated pdf and cdf of competitive models for the second data set

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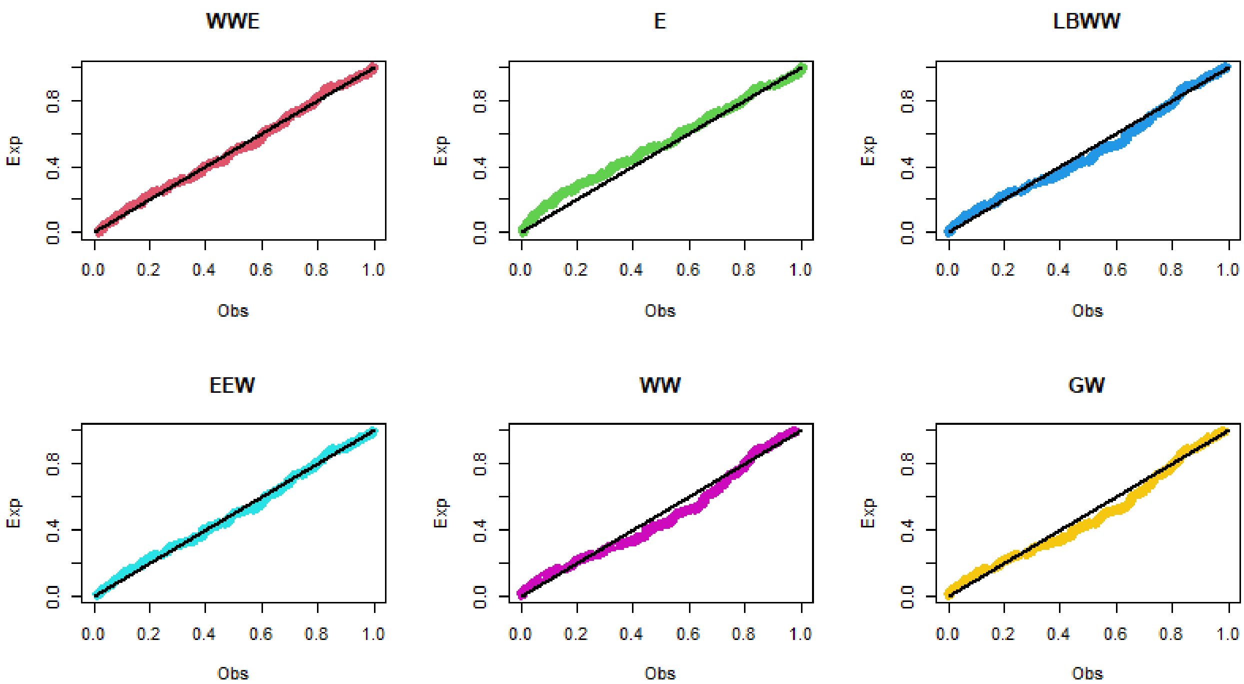

Figure 10. PP plots of the fitted models for the second data set

like oil, coal, and gas for energy, as well as the burning of wood and waste materials and industrial operations like cement manufacturing. As a result, the earth's surface temperature is rising, affecting the climate, sea-level rise, and global agriculture. Real data applications demonstrate that the WWE model frequently gives better fits compared to some other alternative models.

Also from Figures 8 and 10 we can see that the WWE distribution provides a better fit than the other five competitive models for both data sets.

7. Conclusion

In this article, we introduce the weighed Weibull-G family of distributions. Properties of the WW-G are discussed, such as, expressions for the density function, moments, mean deviation, quantile function and order statistics. The maximum likelihood method is employed for estimating the model parameters. More specifically, weighed Weibull generated family covers several new distributions. We wish a broadly statistical application in some area for this new generalization. We used the WWE distribution as one of the models from the WW-G distributions to fit two real data sets that are affected by climate change. The first proposed data collected from renewable technologies in 75 country is based on primary energy calculated by the 'substitution method' which attempts to correct for the inefficiencies in fossil fuel production. The second set of data provides global CO2 emissions per person in 211 country in 2020. This is an example of CO2 emissions coming from the combustion of fossil fuels

REFERENCES

1. Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, 36(1), 106–108.
2. Abdul-Moniem, I. and Diab, L. S. (2018). The length-biased weighted exponentiated Lomax distribution. *International Journal For Research In Mathematics And Statistics*,4(1), 1–14
3. Ahmad, A., Ahmad, S. and Ahmed, A. (2016). Length-biased weighted Lomax distribution: statistical properties and application. *Pakistan Journal of Statistics and Operation Research*, *12*(2), 245-255
4. Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating

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applications. *Journal of Data Science*, 16(4), 723-742.
15. Hassan, A.S. and Nassr, S.G. (2019). Power Lindley-G family. *Annals of Data Science*, 6(2),189–210.
16. Hassan, A.S., Elgarhy, M. and Shakil, M. (2017a). Type II half Logistic family of distributions with applications. *Pakistan Journal of Statistics and Operation Research*, 13(2): 245–264.
17. Hassan, A.S., Hemeda, S.E., Maiti, S.S. and Pramanik, S. (2017b). The generalized additive Weibull-G family of probability distributions. *International Journal of Statistics and Probability*, 6(5): 65–83.
18. Hassan, A.S., Almetwally, E.M., Khaleel, M.A., Nagy, H,F. (2021). Weighted power Lomax distribution and its length biased version: properties and estimation based on censored samples. *Pakistan Journal of Statistics and Operation Research*, 17(2), 343–356
19. Kersey, J., & Oluyede, B. O. (2012). Theoretical properties of the length-biased inverse Weibull distribution. *Involve Journal of Mathematics, 5*, 379–391.
20. Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, 46, 79–88.
21. Nasiru, S. (2015). Another weighted Weibull distribution from Azzalini’s family. *European scientific journal*, 11(9), 134–144.
22. Patil, G.P. and Ord, J. K. (1975), On size-biased sampling and related form-invariant weighted distributions. *Sankhya, Series B*, 38, 48–61.
23. Patil, G.P. and Rao, C. R. (1977).Weighted distributions: a survey of their applications. In Applications of Statistics. *P. R. Krishnaiah* (ed.), 383–405.
24. Rao, C. R. (1965). On discrete distributions arising out of methods of ascertainment, in Classical and Contagious Discrete Distribution. *Patil, G. P. ed. Pergamon Press and Statistical Publishing Society*, Calcutta, 320–332.

families of distributions. *Metron*, **71**: 63–79.
5. Bourguignon, M., Silva, R.B. and Cordeiro, G.M. (2014). The Weibull-G family of probability distributions. *Journal of Data Science*, 12(1), 53–68.
6. Cordeiro, G. M., Alizadeh, M., and Diniz Marinho, P. R. (2016). The type I half-logistic family of distributions. *Journal of Statistical Computation and Simulation*, 86(4), 707–728.
7. Cordeiro, G. M., Alizadeh, M., Ozel, G., Hosseini, B., Ortega, E. M. M., & Altun, E. (2017). The generalized odd log-logistic family of distributions: properties, regression models and applications. *Journal of Statistical Computation and Simulation*, 87 (5), 908–932.
8. Fisher, R. A. (1934). The effects of methods of ascertainment upon the estimation of frequencies. *The Annals of Eugenics, 6*, 13–25
9. Ghitany, M. E., Al-qallaf, F., Al-Mutairi, D. K. and Husain, H. A. (2011), A two parameter weighted Lindley distribution and its applications to survival data. *Mathematics and computers in Simulation*, 81, 1190–1201.
10. Haq, M. and Elgarhy, M. (2018). The odd Frèchet-G family of probability distributions. *Journal of Statistics Applications & Probability*, 7, 185–201.
11. Hassan, A.S. and Elgarhy, M. (2016a). Kumaraswamy Weibull-generated family of distributions with applications. *Advances and Applications in Statistics*, 48(3): 205–239.
12. Hassan, A.S. and Elgarhy, M. (2016b). A new family of exponentiated Weibull-generated distributions. *International Journal of Mathematics and its Applications*, 4(1): 135-148
13. Hassan, A.S. and Hemeda, S.E. (2016). The additive Weibull-G family of probability distributions. *International Journals of Mathematics and Its Applications*, 4: 151–164.
14. Hassan, A.S. and Nassr, S.G. (2018). The inverse Weibull generator of distributions: properties and

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25. Rao, C. R. (1985),Weighted distributions arising out of methods of ascertainment: What population does a sample represent Chapter 24 in Celebration of Statistics, The 151 Centenary Volume, A. C. Atkinson and S E Fienbwg (eds), 543–569.
26. Sen, S, Chandra, N. and Maiti, S.S. (2017). The weighted x-gamma distribution: properties and application. *Journal of Reliability and Statistical Studies*, 10 (1): 43–58

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