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FUZZY SEMI GROUPS

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Abstract

The main aim of this article is to introduce the concept of a sup-hesitant fuzzy ideal, which is a generalization of a hesitant fuzzy ideal and an interval-valued fuzzy ideal, in a ternary semigroup. Some characterizations of a sup-hesitant fuzzy ideal are examined in terms of a fuzzy set, a hesitant fuzzy set, and an interval valued fuzzy set. Further, we discuss the relation between an ideal and a generalization of a characteristic hesitant fuzzy set and a characteristic interval-valued fuzzy set.

Keywords: Ternary semigroup, sup-hesitant fuzzy ideal, Hesitant fuzzy ideal, Interval-valued fuzzy ideal

1. Introduction

Ternary algebraic structures were introduced by Lehmer [1] in 1932, who examined exact ternary algebraic structures called triplexes, which turned out to be ternary groups. Ternary semigroups were first introduced by Stefan Banach, who showed that a ternary semigroup does not necessarily reduce to a semigroup. In 1965, Sioson [2] studied ideal theory in ternary semigroups. In addition, Iampan [3] studied the lateral ideal of a ternary semigroup in 2007. Ideal theory is an important concept for studying ternary semigroups and algebraic structures.

After the concept of a fuzzy set was introduced by Zadeh [4], the ideal theory in a ternary semigroup was extended to fuzzy ideal theory, bipolar fuzzy ideal theory, interval-valued fuzzy ideal theory, and hesitant fuzzy ideal theory in a ternary semigroup. In 2012, Kar and Sarkar [5] introduced a fuzzy left (lateral, right) ideal and fuzzy ideal of a ternary semigroup and used a fuzzy set to characterize a regular (intra-regular) ternary semigroup. In 2015, Ansari and Masmali [6] studied the bipolar (λ, δ)-fuzzy ideal of a ternary semigroup. In 2016, Jun et al. [7] introduced a hesitant fuzzy semigroup with a frontier and studied the hesitant union and hesitant intersection of two hesitant fuzzy semigroups with a frontier. Muhiuddin [8] introduced a hesitant fuzzy G-filter for a residuated lattice and provided some conditions for a hesitant fuzzy filter to be a hesitant fuzzy G-filter. In 2018, Suebsung and Chinram [9] studied an interval-valued fuzzy ideal extension of a ternary semigroup. In 2019, Muhiuddin et al. [10] introduced an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy left (right, lateral) ideal in a ternary semigroup. In addition, in 2020, Talee et al. [11] introduced a hesitant fuzzy ideal and a hesitant fuzzy interior ideal in an ordered Γ-semigroup and characterized a simple ordered Γ-semigroup in terms of a hesitant fuzzy simple ordered Γ-semigroup.

The main aim of this article is to introduce the concept of a sup-hesitant fuzzy ideal of a ternary semigroup, which is a generalization of a hesitant fuzzy ideal and an interval-valued fuzzy ideal in a ternary semigroup. Some characterizations of an sup-hesitant fuzzy ideal are examined in terms of a fuzzy set, a hesitant fuzzy set, and an interval valued fuzzy set. Further, we discuss the relation between an ideal and a generalization of a characteristic hesitant fuzzy set and a characteristic interval-valued fuzzy set.

2. Preliminaries

In the following sections, we introduce some definitions and results that are important for the present study.

By a ternary semigroup, we mean a set $T \neq \emptyset$ with a ternary operation $T \times T \times T \to T$, written as $(t_1, t_2, t_3) \mapsto t_1t_2t_3$ satisfying the identity (for all $t_1, t_2, t_3, t_4, t_5 \in T$)($(t_1t_2t_3)t_4t_5 = t_1(t_2t_3t_4)t_5 = t_1t_2(t_3t_4t_5)$). Throughout this paper, T is represented as a ternary semigroup. Let $X \neq \emptyset$, $Y \neq \emptyset$, and $Z \neq \emptyset$ be subsets of T. We define the subset XYZ of T as follows: $XYZ = \{xyz \mid x \in X, y \in Y, z \in Z\}$. A subset $A \neq \emptyset$ of T is said to be a left (lateral, right) ideal (L(Lt, R)I) of T and $TTA \subseteq A$ ($TAT \subseteq A$, $ATT \subseteq A$). If the subset is an LI, LtI, and RI of T, then it is said to be an ideal (Id) of T.

A fuzzy set (FS) f [4] in set $X \neq \emptyset$ is a mapping from X to the unit segment of the real line [0, 1]. Kar and Sarkar [5] studied an FS in a ternary semigroup and introduced the concepts of a fuzzy left (lateral, right) ideal and a fuzzy ideal of ternary semigroups as follows:

Definition 2.1 [5]

Let f be the FS in T. Then, f is said to be

- (1) a fuzzy left ideal (FLI) of T while (for all $t_1, t_2, t_3 \in T$)($f(t_3) \leq f(t_1t_2t_3)$),
- (2) a fuzzy lateral ideal (FLtI) of T while (for all $t_1, t_2, t_3 \in T$)($f(t_2) \leq f(t_1t_2t_3)$),
- (3) a fuzzy right ideal (FRI) of T while (for all $t_1, t_2, t_3 \in T$)($f(t_1) \leq f(t_1t_2t_3)$), or
- (4) a fuzzy ideal (FI) of T while f is an FLI, an FLtI, and an FRI of T, that is, (for all $t_1, t_2, t_3 \in T$)($\max\{f(t_1), f(t_2), f(t_3)\} \leq f(t_1t_2t_3)$).

Let [[0, 1]] be the set of all closed subintervals of [0, 1]; that is

$[[0,1]]=\{[t^-,t^+] \mid t^-,t^+\in[0,1] \text{ and } t^-\leq t^+\}$.

Let $\hat{t_1}=[t_1^-,t_1^+],\hat{t_2}=[t_2^-,t_2^+]\in[[0,1]]$. We define the operations $\preceq$, $=$, $\prec$, and rmax as follows:

- (1) $\hat{t_1}\preceq\hat{t_2}\Leftrightarrow t_1^-\leq t_2^-,t_1^+\leq t_2^+$,
- (2) $\hat{t_1}=\hat{t_2}\Leftrightarrow t_1^-=t_2^-,t_1^+=t_2^+$,
- (3) $\hat{t_1}\prec\hat{t_2}\Leftrightarrow \hat{t_1}\preceq\hat{t_2},\hat{t_1}\neq\hat{t_2}$,
- (4) $\text{rmax}\{\hat{t_1},\hat{t_2}\}=[\max\{t_1^-,t_2^-\},\max\{t_1^+,t_2^+\}]$.

Let $X \neq \emptyset$ be a set. A mapping $\hat{v}: X \to [[0, 1]]$ is said to be an interval-valued fuzzy set (IvFS) [12] on X, where for any $x \in X$, $\hat{v}(x) = [v^-(x), v^+(x)]$, anything v^- and v^+ are FSs in X such that $v^-(x) \leq v^+(x)$.

For a subset A of X, the characteristic interval-valued fuzzy set CI_A of A on X is defined by

$CI_A: X\to[[0,1]], x\mapsto \begin{cases}\hat{1} & \text{if } x\in A,\\ \hat{0} & \text{otherwise},\end{cases}$

where $\hat{0} = [0, 0]$ and $\hat{1} = [1, 1]$.

Definition 2.2 [9]

Let $\hat{v}$ be an IvFS on T. Then, $\hat{v}$ is said to be

- (1) an interval-valued fuzzy left ideal (IvFLI) of T while (for all $t_1, t_2, t_3 \in T$)($\hat{v}(t_3) \preceq \hat{v}(t_1t_2t_3)$),
- (2) an interval-valued fuzzy lateral ideal (IvFLtI) of T while (for all $t_1, t_2, t_3 \in T$)($\hat{v}(t_2) \preceq \hat{v}(t_1t_2t_3)$),
- (3) an interval-valued fuzzy right ideal (IvFRI) of T while (for all $t_1, t_2, t_3 \in T$)($\hat{v}(t_1) \preceq \hat{v}(t_1t_2t_3)$),
- (4) an interval-valued fuzzy ideal (IvFI) of T while it is an IvFLI, an IvFLtI, and an IvFRI of T, that is, (for all $t_1, t_2, t_3 \in T$)($\text{rmax}\{\hat{v}(t_1), \hat{v}(t_2), \hat{v}(t_3)\} \preceq \hat{v}(t_1t_2t_3)$).

Theorem 2.3 [9]

A subset $A \neq \emptyset$ of T is an Id of T if and only if CI_A is an IvFI of T.

Torra and his colleague [13,14] defined a hesitant fuzzy set (HFS) on a set $X \neq \emptyset$ in terms of a mapping h that, when applied to X, returns a subset of [0, 1], that is, $h: X \to \wp[0, 1]$, where $\wp[0, 1]$ denotes the set of all subsets of [0, 1]. Talee et al. [15] applied the concept of an HFS to a ternary semigroup and introduced the concepts of a hesitant fuzzy left (lateral, right) ideal and a hesitant fuzzy ideal of a ternary semigroup as follows:

Definition 2.4 [15]
Let h be an HFS on T. Then, h is said to be

- (1) a hesitant fuzzy left ideal (HFLI) of T while (for all $t_1, t_2, t_3 \in T)(h(t_3) \subseteq h(t_1t_2t_3))$,
- (2) a hesitant fuzzy lateral ideal (HFLtI) of T while (for all $t_1, t_2, t_3 \in T)(h(t_2) \subseteq h(t_1t_2t_3))$,
- (3) a hesitant fuzzy right ideal (HFRI) of T while (for all $t_1, t_2, t_3 \in T)(h(t_1) \subseteq h(t_1t_2t_3))$,
- (4) a hesitant fuzzy ideal (HFI) of T while it is a HFLI, a HFLtI, and a HFRI of T, that is, (for all $t_1, t_2, t_3 \in T)(h(t_1) \cup h(t_2) \cup h(t_3) \subseteq h(t_1t_2t_3))$.

For a subset A of a set $X \neq \emptyset$, define the characteristic hesitant fuzzy set (CHFS) CH_A of A on X as follows:
$CH_A: X \to P[0,1], x \mapsto \{[0,1] \text{ while } x \in A, \emptyset \text{ otherwise}.$

Theorem 2.5 [15]
A subset $A \neq \emptyset$ of T is an Id of T if and only if CH_A is an HFI of T.
It is well known that an HFS on T is a generalization of the concept of an IvFS on T. In general, we can see that the HFI of T is not an IvFI of T, and an IvFI of T is not an HFI of T, as shown in Example 2.6.

Example 2.6
Consider a ternary semigroup $T = \{-i, 0, i\}$ under the usual multiplication over a complex number.

- (1) Define an HFS h on T by $h(i) = h(-i) = \{0.1, 0.2, 0.3, 0.5\}$ and $h(0) = [0.1, 0.5]$, and we have h as an HFI of T but not an IvFI of T because h is not an IvFS on T.
- (2) Define an IvFS $\hat{v}$ on T by $\hat{v}(-i) = \hat{v}(i) = [0, 0.5]$ and $\hat{v}(0) = [0.5, 1]$, and we have $\hat{v}$ as an IvFI of T but not an HFI of T because
  $\hat{v}(i) \cup \hat{v}(-i) \cup \hat{v}(0) = [0,1] \not\subset [0.5,1] = \hat{v}(0) = \hat{v}((i)(0)(-i))$.
- (3) Define an IvFS g on T by $g(i) = g(-i) = [0, 0.4]$ and $g(0) = [0, 1]$. Then, g is both an HFI and an IvFI of T.

3. Main Results

For $\nabla \in \wp[0, 1]$, define SUP ∇ by
$\mathrm{SUP}\,\nabla = \{\sup \nabla \text{ while } \nabla \neq \emptyset, 0 \text{ otherwise}.$

For an HFS h on X and $\nabla \in \wp[0, 1]$, we define SUP $[h; \nabla]$ as
$\mathrm{SUP}\,[h;\nabla] = \{x \in X \mid \mathrm{SUP}\,h(x) \geq \mathrm{SUP}\,\nabla\}$.

Definition 3.1
Given $\nabla \in \wp[0, 1]$, an HFS h on T is said to be a sup-hesitant fuzzy left (lateral, right) ideal of T related to ∇ (∇-sup-HFL(Lt, R)I of T), whereas the set SUP $[h; \nabla]$ is an L(Lt, R)I of T. If h is a ∇-sup-HFL(Lt, R)I of T for all $\nabla \in \wp[0, 1]$ when SUP $[h; \nabla] \neq \emptyset$, then h is said to be a sup-hesitant fuzzy left (lateral, right) ideal (sup-HFL(Lt, R)I) of T.

Definition 3.2
An HFS h on T is said to be a sup-hesitant fuzzy ideal of T related to ∇ (∇-sup-HFI of T), whereas it is an ∇-sup-HFLI, a ∇-sup-HFLtI, and a ∇-sup-HFRI of T. If h is a ∇-sup-HFI of T for all $\nabla \in \wp[0, 1]$ when SUP $[h; \nabla] \neq \emptyset$, then h is said to be a sup-hesitant fuzzy ideal (sup-HFI) of T.

Lemma 3.3
All IvFL(Lt, R)Is of T are a sup-HFL(Lt, R)I.
Proof
Suppose that $\hat{v}$ is an IvFLI of T and $\nabla \in \wp[0, 1]$ such that SUP $[\hat{v}; \nabla] \neq \emptyset$. Let $a, b \in T$, and let $c \in$ SUP $[\hat{v}; \nabla]$. Then, $\sup \hat{v}(c) \geq$ SUP ∇. Because $\hat{v}$ is an IvFLI of T, we have
$\mathrm{SUP}\,\nabla \leq \sup \hat{v}(c) = v^+(c) \leq v^+(abc) = \sup \hat{v}(abc)$.

Thus, abc ∈ SUP [$\hat{v}$; ∇]. Hence, SUP [$\hat{v}$; ∇] is an LI of T, which indicates that $\hat{v}$ is a ∇-sup-HFLI of T. Therefore, we conclude that $\hat{v}$ is a sup-HFLI of T.
From Lemma 3.3, we obtain Theorem 3.4.

Theorem 3.4
All IvFIs of T are a sup-HFI.
The converses of Lemma 3.3 and Theorem 3.4 are not true, as shown in Example 3.5.

Example 3.5
Consider a ternary semigroup T = {O, A, B, C, D, I} under the usual matrix multiplication, where
O=(0000),A=(1000),B=(0010),C=(0100),D=(0001),I=(1001).
Define an IvFS $\hat{v}$ on T by
$\hat{v}(O)=[0,1]$, $\hat{v}(A)=[0.4,1]$, $\hat{v}(B)=[0.6,1]$,$\hat{v}(C)=\hat{v}(D)=[0.5,1]$, $\hat{v}(I)=\hat{0}$.
Thus, $\hat{v}$ is a sup-HFI of T but not an IvFI of T. Moreover, we know that

- (1) $\hat{v}$ is not an IvFLI of T because $\hat{v}(OAB) = [0, 1] \prec [0.6, 1] = \hat{v}(B)$.
- (2) $\hat{v}$ is not an IvFLtI of T because $\hat{v}(OAB) = [0, 1] \prec [0.4, 1] = \hat{v}(A)$.
- (3) $\hat{v}$ is not an IvFRI of T because $\hat{v}(CBO) = [0, 1] \prec [0.5, 1] = \hat{v}(C)$.

From Lemma 3.3, Theorem 3.4, and Example 3.5, we find that in an arbitrary ternary semigroup, a sup-HFL(Lt, R)I is a generalization of the concept of an IvFL(Lt, R)I, and a sup-HFI is a generalization of the concept of an IvFI.

Lemma 3.6
All HFL(Lt, R)Is of T are a sup-HFL(Lt, R)I.
Proof
Suppose that h is an HFLI of T and ∇ ∈ ℘[0, 1] such that SUP [h; ∇] ≠ ∅. Let a, b ∈ T and c ∈ SUP [h; ∇]. Then, SUP h(c) ≥ SUP ∇. Because h is an HFLI of T, we have h(c) ⊆ h(abc) and thus SUP h(c) ≤ SUP h(abc). Therefore, abc ∈ SUP [h; ∇]. Hence, SUP [h; ∇] is an LI of T, which signifies that h is a ∇-sup-HFLI of T. We thus conclude that h is a sup-HFLI of T.
From Lemma 3.6, we obtain Theorem 3.7.

Theorem 3.7
All HFIs of T are a sup-HFI.
Example 3.8 shows that the converses of Lemmas 3.6 and Theorem 3.7 do not hold.

Example 3.8
Consider a ternary semigroup T = {O, I, X, Y, Z} under the usual matrix multiplication, where
O=(000000000), I=(100010001), X=(000010000),Y=(000000100),Z=(001000000).

Define an HFS h on T as
h(O)={0,1}, h(X)=[0,1], h(Y)=h(Z)=[0,1),h(I)=∅.
Thus, h is a sup-HFI of T, but not an HFI of T. Moreover, we know that

- (1) h is not an HFLI of T because h(X) = [0, 1] ⊃ {0, 1} = h(OYX).
- (2) h is not an HFLtI of T because h(X) = [0, 1] ⊃ {0, 1} = h(OXI).
- (3) h is not an HFRI of T because h(X) = [0, 1] ⊃ {0, 1} = h(XOZ).

From Lemma 3.6, Theorem 3.7, and Example 3.8, we find that in an arbitrary ternary semigroup, a sup-HFL(Lt, R)I is a generalization of the concept of an HFL(Lt, R)I, and a sup-HFI is a generalization of the concept of an HFI.
Let h be an HFS on T, and define the FS F_h in T as
$F_h: T \to [0,1], x \mapsto \mathrm{SUP}\ h(x)$.

The following lemma characterizes the sup-types of HFSs on T by FS F_h.

Lemma 3.9
An HFS h on T is a sup-HFL(Lt, R)I of T if and only if F_h is an FL(Lt, R)I of T.
Proof
Suppose that h is an sup-HFLI of T. Let a, b, c ∈ T, and let ∇ = h(c). Then, c ∈ SUP [h; ∇]. Thus, h is a ∇-sup-HFLI of T, which indicates that SUP [h; ∇] is an LI of T. Hence, abc ∈ SUP [h; ∇] and thus Fh(abc)=SUP h(abc)≥SUP ∇=SUP h(c)=Fh(c).

Therefore, F_h is an FLI of T.
Conversely, suppose that F_h is an FLI of T and ∇ ∈ ℘[0, 1] such that SUP [h; ∇] ≠ ∅. Let a, b ∈ T and c ∈ SUP [h; ∇]. Then,
SUP h(abc)=Fh(abc)≥Fh(c)=SUP h(c)≥SUP ∇,
and it is implied that abc ∈ SUP [h; ∇]. Hence, SUP [h; ∇] is an LI of T; that is, h is a ∇-sup-HFLI of T. Therefore, we conclude that h is a sup-HFLI of T.
Let h be an HFS on T and ∇ ∈ ℘[0, 1], and we define the HFS H(h; ∇) on T as
(for all x∈T)(H(h;∇)(x)={t∈∇|SUP h(x)≥t}).

We then denote $H(h;\bigcup_{x\in T} h(x))$ by H_h and H(h; [0, 1]) by I_h. Then, I_h is an IvFS on T.

Remark 3.10
If h is an HFS on T, then $h(x) \subseteq H_h(x) \subseteq I_h(x)$ and SUP h(x) = SUPH_h(x) = supI_h(x) for all x ∈ T.
Now, we study sup-types of HFSs on T using the HFS H(h; ∇) and the IvFS I_h.

Lemma 3.11
An HFS h on T is a sup-HFL(Lt, R)I of T if and only if H(h; ∇) is a HFL(Lt, R)I of T for all ∇ ∈ ℘[0, 1].

Proof
Suppose that h is a sup-HFLI of T and ∇ ∈ ℘[0, 1]. Let a, b, c ∈ T. If H(h; ∇)(c) is empty, then H(h; ∇)(c) ⊆ H(h; ∇)(abc). In addition, let t ∈ H(h; ∇)(c). Then, t ∈ ∇, SUP h(c) ≥ t, and c ∈ SUP [h; h(c)]. Because h is a sup-HFLI of T, we have SUP [h; h(c)] as an LI of T. Hence, abc ∈ SUP [h; h(c)], which indicates that SUP h(abc) ≥ SUP h(c) ≥ t. Thus, t ∈ H(h; ∇)(abc). Therefore, H(h; ∇)(c) ⊆ H(h; ∇)(abc). Consequently, H(h; ∇) is an HFLI of T.
Conversely, suppose that H(h; ∇) is an HFLI of T for all ∇ ∈ ℘[0, 1]. Let a, b, c ∈ T and ∇ ∈ ℘[0, 1] exist such that c ∈ SUP [h; ∇]. Then, H(h; ∇)(c) = ∇, and by assumption, we have ∇ = H(h; ∇)(c) ⊆ H(h; ∇)(abc). Thus, SUP h(abc) ≥ SUP ∇, and it is implied that abc ∈ SUP [h; ∇]. Hence, SUP [h; ∇] is an LI of T; that is, h is a ∇-sup-HFLI of T. Therefore, we conclude that h is a sup-HFLI of T.

Theorem 3.12
For an HFS h on T, the following statements are equivalent.
- (1) h is a sup-HFL(Lt, R)I of T.
- (2) H_h is an HFL(Lt, R)I of T.
- (3) H_h is a sup-HFL(Lt, R)I of T.
- (4) I_h is an IvFL(Lt, R)I of T.
- (5) I_h is a sup-HFL(Lt, R)I of T.
- (6) I_h is an HFL(Lt, R)I of T.

Proof
(1) ⇒ (2) and (1) ⇒ (6). These follow from Lemma 3.11.

(2) ⇒ (3) and (6) ⇒ (5). These follow from Lemma 3.6.
(4) ⇒ (5). This follows from Lemma 3.3.
(3) ⇒ (1). Suppose that H_h is an sup-HFLI of T and $\nabla \in \wp[0, 1]$ such that SUP $[h; \nabla] \neq \emptyset$. Let a, b ∈ T and c ∈ SUP $[h; \nabla]$. Based on Remark 3.10, we have $\text{SUP}H_h(c)$ = SUP h(c) ≥ SUP ∇ and thus c ∈ SUP $[H_h; \nabla]$. We assume that SUP $[H_h; \nabla]$ is an LI of T, and then abc ∈ SUP $[H_h; \nabla]$. By Remark 3.10 again, we can see that SUP h(abc) = $\text{SUP}H_h$ (abc) ≥ SUP ∇, which signifies that abc ∈ SUP [h; ∇]. Hence, SUP [h; ∇] is an LI of T; that is, h is a ∇-sup-HFLI of T. We therefore conclude that h is a sup-HFLI of T.
(1) ⇒ (4). Suppose that h is a sup-HFLI of T and a, b, c ∈ T. Then, c ∈ SUP [h; h(c)], and therefore by assumption we have abc ∈ SUP [h; h(c)]. Thus, SUP h(c) ≤ SUP h(abc), and therefore $I_h(c)$ = [0, SUP h(c)] ⪯ [0, SUP h(abc)] = $I_h(abc)$. Hence, I_h is an IvFLI of T.
(5) ⇒ (1). Let I_h be an sup-HFLI of T and $\nabla \in \wp[0, 1]$ such that SUP [h; ∇] ≠ ∅. Let a, b ∈ T and c ∈ SUP [h; ∇]. By Remark 3.10, we have sup $I_h(c)$ = SUPh(c) ≥ SUP ∇, and thus c ∈ SUP $[I_h; \nabla]$. We assume that abc ∈ SUP $[I_h; \nabla]$. By Remark 3.10, we obtain SUP h(abc) = $\text{sup}I_h$ (abc) ≥ SUP ∇, which indicates that abc ∈ SUP [h; ∇]. Hence, SUP[h; ∇] is an LI of T, which signifies that h is a ∇-sup-HFLI of T. Therefore, we conclude that h is a sup-HFLI of T.

From Lemma 3.9 and Theorem 3.12, we obtain Theorem 3.13.

Theorem 3.13
For an HFS h on T, the following statements are equivalent.
- (1) h is a sup-HFI of T.
- (2) (for all a, b, c ∈ T)(SUP h(abc) ≥ max{SUP h(a), SUP h(b), SUP h(c)}).
- (3) F_h is an FI of T.
- (4) H_h is an HFI of T.
- (5) H_h is a sup-HFI of T.
- (6) I_h is an IvFI of T.
- (7) I_h is a sup-HFI of T.
- (8) I_h is an HFI of T.

For a subset A of T and ∇, Ω ∈ ℘[0, 1] with SUP ∇ < SUP Ω, we define a map H(∇,Ω)A as follows:
H(∇,Ω)A:T→P[0,1],x↦{Ω∇while x∈A,otherwise.

Then, H(∇,Ω)A is an HFS on T, which is said to be a sup (∇, Ω)-characteristic hesitant fuzzy set (sup (∇, Ω)-CHFS) of A of T. In addition, sup (∇, Ω)-CHFS with ∇ = ∅ and Ω = [0, 1] is the CHFS of A, that is, H(∅,[0,1])A=CHA. Moreover, sup (∇, Ω)-CHFS with ∇ = $\hat{0}$ and Ω = $\hat{1}$ is the CIvFS of A, that is, H(0ˆ,1ˆ)A=CIA.

Theorem 3.14
Let a subset A ≠ ∅ of T and ∇, Ω ∈ ℘[0, 1] exist such that SUP ∇ < SUP Ω. Then, A is an Id of T if and only if H(∇,Ω)A is an sup-HFI of T.

Proof
Suppose that there exist a, b, c ∈ T such that
SUP H(∇,Ω)A(abc)<max{SUP H(∇,Ω)A(a), SUP H(∇,Ω)A(b), SUP H(∇,Ω)A(c)}. Then,
H(∇,Ω)A(a)=Ω, H(∇,Ω)A(b)=Ω, or H(∇,Ω)A(c)=Ω, which signifies that a ∈ A, b ∈ A, or c ∈ A. Because A is an Id of T, we have abc ∈ A and H(∇,Ω)A(abc)=Ω. Thus,
SUP H(∇,Ω)A(abc)=max{SUP H(∇,Ω)A(a),SUP H(∇,Ω)A(b),SUP H(∇,Ω)A(c)}
is a contradiction. Hence,
SUP H(∇,Ω)A(abc)≥max{SUP H(∇,Ω)A(a),SUP H(∇,Ω)A(b),SUP H(∇,Ω)A(c)}
for all a, b, c ∈ T, and by Theorem 3.13, we have H(∇,Ω)A being a sup-HFI of T.

Conversely, let $a \in A$ and $x, y \in T$. Then $H(\nabla,\Omega)A(a)=\Omega$. Because $H(\nabla,\Omega)A$ is a sup-HFI of T, and by Theorem 3.13, we have

$SUP\ H(\nabla,\Omega)A(axy) \geq \max\{SUP\ H(\nabla,\Omega)A(a), SUP\ H(\nabla,\Omega)A(x), SUP\ H(\nabla,\Omega)A(y)\}$, $SUP\ H(\nabla,\Omega)A(xay) \geq \max\{SUP\ H(\nabla,\Omega)A(a), SUP\ H(\nabla,\Omega)A(x), SUP\ H(\nabla,\Omega)A(y)\}$,

and

$SUP\ H(\nabla,\Omega)A(xya) \geq \max\{SUP\ H(\nabla,\Omega)A(a), SUP\ H(\nabla,\Omega)A(x), SUP\ H(\nabla,\Omega)A(y)\} = SUP\ \Omega$.

Thus,

$SUP\ H(\nabla,\Omega)A(axy)=SUP\ H(\nabla,\Omega)A(xay)=SUP\ H(\nabla,\Omega)A(xya)=SUP\ \Omega$,

which indicates that axy, xay, xya $\in$ A. Hence, A is the Id of T.

From Theorems 2.3, 2.5, 3.4, 3.7, and 3.14, we obtain Theorem 3.15.

Theorem 3.15

For a subset $A \neq \emptyset$ of T, the following statements are equivalent.

- (1) A is an Id of T.
- (2) CI_A is an IvFI of T.
- (3) CI_A is a sup-HFI of T.
- (4) CH_A is an HFI of T.
- (5) CH_A is a sup-HFI of T.
- (6) $H(\nabla,\Omega)A$ is a sup-HFI of T for all $\nabla, \Omega \in P[0, 1]$ with $SUP\ \nabla < SUP\ \Omega$

4. Conclusion

In this paper, we introduced the concept of a sup-HFI in a ternary semigroup, which is a generalization of an HFI and an IvFI in a ternary semigroup, and examined some characterizations of a sup-HFI in terms of an FS, an HFS, and an IvFS. Further, we discussed the relation between an Id and the generalizations of CHFSs and CIvFSs. As important study results, we found that the following statements are all equivalent in a ternary semigroup T: A subset A is an Id, CI_A is an IvFI, CI_A is a sup-HFI, CH_A is an HFI, and CH_A is a sup-HFI.

In the future, we will study a sup-HFI in a Γ semigroup and examine some characterizations of a sup-HFI in terms of an FS, an HFS, and an IvFS.

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