2-EDGE DISTANCE-BALANCED GRAPHS

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Abstract

In a graph A, for each two arbitrary vertices g, h with d(g, v) = 2, $|M_{g_2h}^A| = m_{g_2h}^A$ is introduced the number of edges of A that are closer to g than to h. We say A is a 2edge distance-balanced graph if we have $m_{g_2h}^A = m_{h_2g}^A$. In this article, we verify the concept of these graphs and present a method to recognize k-edge distance-balanced graphs for k = 2,3 using existence of either even or odd cycles. Moreover, we investigate situations under which the cartesian and lexicographic products lead to 2edge distance-balanced graphs. In some subdivision-related graphs 2-edge distancebalanced property is verified. Mathematics Subject Classification: 05C12, 05C25

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1 INTRODUCTION

The notion of graph is a pivotal tool to make use of the modeling of the phenomena and it is taken into consideration in many studies in a recent decades. One of the optimal uses of graphs theory is to classify graphs based on discriminating quality. This phenomenon can be best observed in distance-balanced graphs has been determined by [10]. Also, it is investigated in some papers, we refer the reader to ([1],[2],[3],[5],[8],[11]-[15]) and references therein.

Let *A* be a connected, finite and undirected graph throughout of this paper, in which V(A)is its vertex set and E(A) is its edge set. In a graph *A*, the distance between each pair of vertices $g, h \in V(A)$ is introduced the number of edges in the least distance joining them and it is indicated by $d_A(g, h)$ (see [3, 15]. For every two desired edges f = gh, $\hat{f} = \hat{g}\hat{h}$, the distance between *f* and \hat{f} is introduced via:

 $\begin{aligned} d_A(f, f) &= \min\{d_A(g, f), d_A(h, f)\} \\ &= \min\{d_A(g, g), d_A(g, h), d_A(h, g), d_A(h, h)\}. \\ \text{Set} & m_g(f) = |M_g(f)| = |\{f \in E(A) | d_A(g, f) < d_A(h, f)\}| \end{aligned}$

$$\begin{split} m_{v}(f) &= |M_{h}(f)| = \\ \left| \left\{ \hat{f} \in E(A) \middle| d_{A}(h, \hat{f}) < d_{A}(g, \hat{f}) \right\} \right| \\ \text{and} \\ m_{0}(f) &= |M_{0}(f)| = \\ \left| \left\{ \hat{f} \in E(G) \middle| d_{A}(g, \hat{f}) = d_{A}(h, \hat{f}) \right\} \right|. \end{split}$$
Presume that $f = gh \in E(A)$. For every two integers *i*, *j*, we consider: $\hat{D}_{i}^{i}(e) = \{ \hat{f} \in E(A) | d_{A}(\hat{f}, g) = i. d_{A}(\hat{f}, h) = j \}.$ A "distance partition" of E(A) is concluded by sets $\hat{D}_i^i(f)$ due to the edge f = gh. Merely the sets $\hat{D}_i^{i-1}(f)$, $\hat{D}_i^i(f)$ and, $\hat{D}_{i-1}^i(f)$, for every $(1 \le 1)$ $i \leq d$) might be nonempty according to the triangle inequality (d is the diameter of the graph A). As well as $\hat{D}_0^0(e) = \phi$. For an edge f = gh of A we denote $n_a^A(f) =$ $|W_{q,h}^{A}| = |\{a \in V(A) | d_{A}(a,g) < d_{A}(a,h)\}|$ Analogously, we would define $n_h^A(f) = |W_{h,a}^A|$. Graph A is named distance-balanced (DB) whenever for an edge f = gh of A we have $n_a^A(f) = n_h^A(f)$ [10]. Graph *A* is called edge-distance-balanced (EDB) whenever for an edge f = gh of A there is a positive integer γ , so that $m_q^A(f) = m_h^A(f) = \gamma$ [16].

In Section 2, it is introduced an extended version of the concept of edge distance-balanced, that is called 2-edge distanced-balanced. We study 2edge distance-balanced graphs in the framework of cartesian and lexicographic products of two connected graphs in Section 3. Finally, in Section 4, it is investigated 2-edge distancebalanced property in some subdivision-related graphs.

2 Specification of k-edge distance-balanced graphs (k = 2, 3)

In this segment, we would complete results of ndistance-balanced graphs that it has already expressed by [6] for k-edge distance-balanced graphs for k = 2,3 and present a method for classification such graphs.

Definition 21 Let A be a graph. It is denominated 2-edge distance-balanced (briefly 2 - EDB) if and only if for each two of vertices $g, h \in V(A)$ with d(g, h) = 2, it holds that $|M_{g_2h}^A| = |M_{h_2g}^A|$, in which

$$m_{g_2h}^{\bar{A}} = |M_{g_2h}^{\bar{A}}| =$$

$$\begin{split} m_{g_{2}h}^{..} &- |m_{g_{2}m}| \\ |\{f \in E(A) | d(f,g) < d(f,h)\}|. \\ & m_{h_{2}g}^{A} = |M_{h_{2}g}^{A}| = \end{split}$$

 $|\{f \in E(A) | d(f,h) < d(f,g)\}|.$ Also, consider the concept $m_{0g_2h}^A = |M_{0g_2h}^A| =$ $\{f \in E(A) | d(f,h) = d(f,g)\}.$

The notion $g_{\underline{2}}h$ is equvalent a path with length 2. Obviously, a distance partition for E(A) is formed by $M_{g_2h}^A$. $M_{h_2g}^A$ and $M_{0g_2h}^A$.

Example 22 Star graphs S_k (with k > 1), Wheel graphs W_n (with n > 4), Frienship graphs F_n (with n > 1) are 2-EDB graphs. Complete bipartite graph $K_{m,n}$ is 2-EDB but without edge distance-balanced property.

Definition 23 The total distance $D^{A}(g)$ of g, for a vertex g of A is denoted

$$D^{A}(g) = \sum_{f \in E(A)} d^{A}(g, f) = d(g, E(A))$$

Whenever A is explicit from the text we will write d(q, f) and D(q) in lieu of $d^A(q, f)$ and $D^{A}(q)$, respectively.

Definition 24 The property (Δ_n) in a connected graph A, for vertices $g, h \in V(A)$ with d(g,h) = n and $f \in M_{g_nh}^A$ is introduced as the shortest way between f and h that is, (W_{fh}) so that it does not include the shortest way between f and h that is, (W_{fh}) .

Theorem 1 Presume that A is a connected graph. Then under one of the below conditions, A is 2-EDB if and only if for each two of vertices $g, h \in V(A)$ with d(g, h) = 2, $D^A(g) =$ $D^A(h)$ holds.

(i) A does not have any odd cycle, that is, A is bipartite.

(ii) A does not have any even cycle, but has the property (Δ_2) .

Proof Since condition (i) holds, we assume that g, h are two vertices in A, in which d(g, h) = 2. Then $D^A(g) = D^A(h)$ can be shown as follows $\sum_{f \in M_{g_2h}^A} d(g, f) + \sum_{e \in M_{h_2g}^A} d(g, f) +$ $\sum_{f \in M_0^G_{g_2h}} d(g, f) =$ $\sum_{f \in M_{g_2h}^A} d(h, f) + \sum_{f \in M_{h_2g}^A} d(h, f) +$ $\sum_{f\in M^A_{0,g_2h}}d(h,f).$ which concludes that

$$\sum_{f \in M^A_{g_{\underline{2}}h}} (d(g, f) - d(h, f)) =$$
$$\sum_{f \in M^A_{h_2g}} (d(h, f) - d(g, f)). \tag{1}$$

For any $f \in M_{g_2h}^A$ we have d(g, f) - d(h, f) =-2. To prove it, we consider that $f \in M_{g_2h}^A$ and d(f,g) = l(P) so that P is the shortest path between f and g.

Hence,

 $l(P) = d(f,g) < d(f,h) \le l(P) + 2.$ It implies that d(f,h) = l(P) + 1 or l(p) + 2. So long as d(f,h) = l(P) + 1, it can be inferred that there exists an odd cycle having length 2l(P) + 1or 2l(P) + 3. Hence, it contradicts and the claim follows. In the same way, it attains d(h, f) – d(g, f) = -2 for each $f \in M_{h_2g}^A$. Now by (1) we obtain

$$\sum_{f \in M^{A}_{g_{2}h}}(-2) = \sum_{f \in M^{A}_{h_{2}g}}(-2).$$

Thus, it is right if and only if A is 2-EDB. Hence, under condition (i) the proof is completed. For next condition, we consider that A does not have any even cycle and has the property (Δ_2) . Since d(f,g) = l(P), for each edge $f \in M_{g_2h}^A$ then we obtain $d(f,h) \in \{l(P) + 1, l(P) + 2\}$. Since d(f,h) = l(P) + 2, then a path R between f and h with length l(P) + 2 exists. It is seen that there is an even cycle with length 2l(P) + 4 or 2l(P) +2 in A that is a contradiction. Thus,

$$\begin{split} & \sum_{f \in M_{g_2h}^A} \left(d(g, f) - d(h, f) \right) = \\ & \sum_{f \in M_{h_2g}^A} \left(d(h, f) - d(g, f) \right). \\ & \text{then we attain } \sum_{f \in M_{g_2h}^A} (-1) = \sum_{f \in M_{h_2g}^A} (-1) . \\ & \text{The proof ends.} \end{split}$$

Theorem 1 matters to classify a group of the edge distance-balanced graphs with specified properties.

Corollary 25 Every bipartite and 2-EDB graph is edge distance-balanced.

In the next theorem, using the notion of total distance, we present an analogous condition and a procedure to recognize edge distance-balanced graphs with d(g, h) = 3 which are called 3-EDB graphs.

Theorem 2 If *A* be a connected graph having the property (Δ_3) without any even cycle, then *A* is 3-EDB if and only if for each two of vertices $g, h \in V(A)$ with d(g, h) = 3, it holds $D^A(g) = D^A(h)$.

Proof Based on the proof of Theorem 1, we assert that

$$\begin{split} & \sum_{f \in M_{g_3h}^A} \left(d(g,f) - d(h,f) \right) = \\ & \sum_{f \in M_{h_3g}^A} \left(d(h,f) - d(g,f) \right) \\ & \text{and it follows } \sum_{f \in M_{g_3h}^A} (-2) = \sum_{f \in M_{h_3g}^A} (-2) \; . \\ & \text{Let } f \in M_{g_3h}^A \; \text{and} \; d(f,g) = l(R) \; \text{such that } R \\ & \text{is the shortest path between } f \; \text{and } g. \; \text{If } l(R) = \\ & d(g,f) < d(h,f) \leq l(R) + 3, \; \text{then } d(h,f) \; \text{is } \\ & l(R) + 1 \; \text{or } l(R) + 2 \; \text{and or } l(R) + 3. \; \text{Suppose that } m \; \text{and } n \; \text{are in the path between } g \; \text{and } h. \\ & \text{We \; call } l(Q) = d(f,m) \; , \; l(S) = d(f,n) \; \text{and} \end{split}$$

l(P) = d(f,h). Since d(h, f) = l(R) + 1, then concerning the property (Δ_3) and with paths Q, Pand S we obtain an even cycle with 2l(R), 2l(R) + 2 or 2l(R) + 4, respectively. Hence it contradicts. Analogously, since d(f,h) = l(R) +3, then applying the paths Q, S or P an even cycle is observed with length 2l(R) + 2, 2l(R) + 4 or 2l(R) + 6, respectively which makes a contradiction, as well as. So the assertion is proved. Therefore, the result follows.

3 2-edge distance-balanced graphs and product graphs

We would now investigate situations in which the *Cartesian product* leads to 2-EDB graphs. We mention that such product graphs, formed by graphs *A* and *B*, its vertex set is $(A \Box B) = V(A) \times V(B)$. Consider that (a_1, b_1) and (a_2, b_2) are distinct vertices in $V(A \Box B)$. In the Cartesian product $A \Box B$, if two vertices (a_1, b_1) and (a_2, b_2) are coincident in one coordinate and adjacent in the other coordinate, then they are adjacent, that is, $a_1 = a_2$ and $b_1b_2 \in E(B)$, or $b_1 = b_2$ and $a_1a_2 \in E(A)$. Obviously, for vertices we have:

$$\begin{aligned} &d_{A \square B}((a_1, b_1), (a_2, b_2)) = d_A(a_1, a_2) + \\ &d_B(b_1, b_2) \;. \end{aligned}$$

For edges we attain:

 $\begin{aligned} & d_{A \Box B}((a, b)(a_1, b_1), (\acute{a}, \acute{b})(\acute{a}_1, \acute{b}_1)) = \\ & \min\{d_{A \Box B}((a, b), (\acute{a}, \acute{b})), d_{A \Box B}((a, b), (\acute{a}_1, \acute{b}_1)), d_{A \Box B}((a_1, b_1), (\acute{a}, \acute{b})), d_{A \Box B}((a_1, b_1)), \\ & (\acute{a}_1, \acute{b}_1))\} = \\ & \min\{d_A(a, \acute{a}) + d_B(b, \acute{b}), d_A(a, \acute{a}_1) + d_B(b, \acute{b}_1), d_A(a_1, \acute{a}) + d_B(b_1, \acute{b}), d_A(a_1, \acute{a}_1) + \\ & d_B(b_1, \acute{b}_1)\}. \end{aligned}$

Theorem 3 If *A* and *B* are graphs, then $A \square B$ is 2-EDB if and only if both *A* and *B* are 2-EDB and 2-DB.

Proof Consider that a_1 , a_2 are two adjacent vertices in A with $d(a_1, a_2) = 2$ and also b_1, b_2 are two vertices in B with $d(b_1, b_2) = 2$. Let (a_1, b_1) , (a_2, b_1) and $(a_1, b_2) \in V(A \square B)$. Then it is clearly seen that

$$d((a_1, b_1), (a_2, b_1) = 2, d((a_1, b_1), (a_1, b_2))$$

= 2 and d((a_2, b_1), (a_1, b_2))
= 2.

We see that

$$\begin{split} M_{(a_1,b_1)_{\underline{2}}(a_2,b_2)} &= \{(a,b)(\acute{a},\acute{b}) \in E(A \Box B) | a\acute{a} \\ &\in E(A), b = \acute{b} \text{ or } b\acute{b} \in E(B), a \\ &= \acute{a}. \\ \min\{d_A(a,a_1), d_A(\acute{a},a_1)\} < \\ \min\{d_A(a,a_2), d_A(\acute{a},a_2)\}\}. \quad (2) \\ \text{and hence applying (2) and by [16. Theorem 2.1)} \\ \text{we conclude that:} \\ m_{(a_1,b_1)_{\underline{2}}(a_2,b_2)}^{A \Box B} = m_{a_1\underline{2}a_2}^A \cdot |V(B)| + n_{a_1\underline{2}a_2}^A \cdot \\ |E(B)|. \quad (3) \\ \text{Similar to this process we obtain} \\ m_{(a_2,a_2)_{\underline{2}}(a_1,b_1)}^{A \Box B} = m_{a_1\underline{2}a_2}^A \cdot |V(B)| + n_{a_1\underline{2}a_2}^A \cdot \\ |E(B)|, \quad (4) \end{split}$$

$$m_{(a_1,b_1)_2(a_1,b_2)}^{A\square B} = m_{b_1 \underline{2} b_2}^B \cdot |V(A)| + n_{b_1 \underline{2} b_2}^B \cdot |E(A)|, \qquad (5)$$

$$m_{(a_1,b_2)_2(a_1,b_1)}^{A\square B} = m_{b_2 \underline{2} b_1}^B \cdot |V(A)| + n_{b_2 \underline{2} b_1}^B \cdot |E(A)|. \qquad (6)$$
Now, suppose that A and B are 2-EDB and 2-DB graphs. By (3) and (4) we have:
$$m_{a_1 \underline{2} a_2}^A \cdot |V(B)| + n_{a_1 \underline{2} a_2}^A \cdot |E(B)| = m_{a_2 \underline{2} a_1}^A \cdot |V(B)| + n_{a_2 \underline{2} a_1}^A$$

therefore

 $m_{(a_1,b_1)_2(a_2,b_2)}^{A\square B} = m_{(a_2,b_2)_2(a_1,b_1)}^{A\square B}.$ By analogy, using (5) and (6), and we know that B is 2-EDB and 2-DB, we conclude that $m_{(a_1,b_1)_2(a_1,b_2)}^{A\square B} = m_{(a_1,b_2)_2(a_1,b_1)}^{A\square B}.$ and therefore $A\square B$ is 2-EDB. For converse, consider that $A\square B$ is 2-EDB,

then using (3) and (4) we observe that

 $\cdot |E(B)|.$

And for edges we have:

$$\begin{split} m^{A \Box B}_{(a_1,b_1)_{\underline{2}}(a_2,b_2)} &= m^{A \Box B}_{(a_2,b_2)_{\underline{2}}(a_1,b_1)} \Longrightarrow \\ m^A_{a_1\underline{2}a_2} \cdot |V(B)| + n^A_{a_1\underline{2}a_2} \cdot |E(B)| \\ &= m^A_{a_2\underline{2}a_1} \cdot |V(B)| + n^A_{a_2\underline{2}a_1} \\ \cdot |E(B)|. \end{split}$$

therefore *A* is 2-EDB and 2-DB. Similarly (5) and (6) yield that *B* is 2-EDB and 2-DB. The proof is completed. \Box

Here, we would define the lexicographic product graphs. The *lexicographic product* A[B] of two graphs A and B is the graph that $V(A[B]) = V(A) \times V(B)$ is its vertex set and two vertices $(a_1, b_1), (a_2, b_2)$ are adjacent if $a_1 a_2 \in E(A)$ or if $a_1 = a_2$ and $b_1 b_2 \in E(B)$ [9, p.22]. Since A is a connected non-trivial graph, then it is easily seen for vertices that

$$d_{A[B]}((a_1, b_1), (a_2, b_2)) \\ = \begin{cases} d_A(a_1, a_2) & if a_1 \neq a_2 \\ \min\{2, d_B(b_1, b_2)\} & if a_1 = a_2. \end{cases}$$

$$\begin{aligned} & d_{A[B]}((a,b)(a_1,b_1),(\acute{a},\acute{b})(\acute{a}_1,\acute{b}_1)) = \\ & \min \begin{cases} d_A(a,\acute{a}) & if \ a \neq \acute{a}, \ \min\{2,d_B(b,\acute{b})\} & if \ a = \acute{a} \\ d_A(a,\acute{a}_1) & if \ a \neq \acute{a}_1, \ \min\{2,d_B(b,\acute{b}_1)\} & if \ a = \acute{a}_1 \\ d_A(a_1,\acute{a}) & if \ a_1 \neq \acute{a}, \ \min\{2,d_B(b_1,\acute{b})\} & if \ a_1 = \acute{a} \\ d_A(a_1,\acute{a}_1) & if \ a_1 \neq \acute{a}_1, \ \min\{2,d_B(b_1,\acute{b}_1)\} & if \ a_1 = \acute{a}_1 \end{cases} \end{aligned} \right\}.$$

In the following theorem, it is proved that the lexicographic product A[B] of graphs A and B is 2-EDB if and only if A is 2-EDB and B is locally regular. At first, we state the definition of locally regular graphs.

Definition 31 The graph A is locally regular according to n (briefly n-locally regular) if it holds

 $\forall g, h \in V(A), \quad d(g,h) = n \Rightarrow deg(g) = deg(h) \ [6].$

Theorem 4 Presume that A and B are graphs. Then the graph A[B] is 2-EDB if and only if A is 2-EDB and B is locally regular.

Proof For the beginning, let the graph A[B] be 2-EDB and $(a_1, b_1), (a_1, b_2) \in V(A[B]), b_1 \neq b_2$ with $d((a_1, b_1), (a_1, b_2)) = 2$. Then $b_1b_2 \notin E(A)$. For edge $(\dot{a}_1, \dot{h}_1)(\dot{a}_2, \dot{b}_2) \in E(A[B])$ according to the definition of distance between an edge and a vertex, we obtain the following consequence

$$d_{A[B]}((\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{1})) < d_{A[B]}((\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{2})) \Rightarrow$$

$$\min\{d_{A[B]}((\dot{a}_{1}, \dot{b}_{1}), (a_{1}, b_{1})), d_{A[B]}((\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{1}))\} <$$

$$\min\{d_{A[B]}((\dot{a}_{1}, \dot{b}_{1}), (a_{1}, b_{2})), d_{A[B]}((\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{2}))\} \Rightarrow$$

$$\min\{d_{A[B]}((\dot{a}_{1}, \dot{b}_{1}), (a_{1}, b_{2})), d_{A[B]}((\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{2}))\} \Rightarrow$$

$$\begin{cases} \dot{a}_{1} = a_{1}, \quad \dot{b}_{1}b_{1} \in E(B), \dot{b}_{1}b_{2} \notin E(B) \\ or \\ \dot{a}_{2} = a_{1}, \quad \dot{b}_{2}b_{1} \in E(B), \dot{b}_{2}b_{2} \notin E(B) . \end{cases}$$

$$(7)$$
Similarly,
$$d_{A[B]}((\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{2})) < d_{A[B]}((\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{1})) \Rightarrow$$

$$\begin{aligned}
\min\{d_{A[B]}\left((\dot{a}_{1},\dot{b}_{1}),(a_{1},b_{2})\right),d_{A[B]}\left((\dot{a}_{2},\dot{b}_{2}),(a_{1},b_{2})\right)\} < \\
\min\{d_{A[B]}\left((\dot{a}_{1},\dot{b}_{1}),(a_{1},b_{1})\right),d_{A[B]}\left((\dot{a}_{2},\dot{b}_{2}),(a_{1},b_{1})\right)\} \Rightarrow \\
\begin{cases}
\dot{a}_{1} = a_{1}, & \dot{b}_{1}b_{2} \in E(B), & \dot{b}_{1}b_{1} \notin E(B) \\
& or \\
\dot{a}_{2} = a_{1}, & \dot{b}_{2}b_{2} \in E(B), & \dot{b}_{2}b_{1} \notin E(B).
\end{aligned}$$
(8)

 $(a_2 = a_1, b_2b_2 \in E(B), b_2b_1 \notin E(B).$ Since A[B] is 2-EDB by equalities (7) and (8) it follows that, $|\{\dot{b}_1|\dot{b}_1b_1 \in E(A)\}| = |\{\dot{b}_1|\dot{b}_1b_2 \in E(A)\}|,$

and

$$|\{\dot{b}_2|\dot{b}_2b_1 \in E(A)\}| = |\{\dot{b}_2|\dot{b}_2b_2 \in E(A)\}|.$$

If $b_1b_2 \notin E(A)$, then the above implication follows that any non-adjacent vertices of *B* have the same degree. Hence *B* is a locally regular graph.

Now suppose that $(a_1, b_1), (a_2, b_2) \in V(A[B])$ and $(\dot{a}_1, \dot{b}_1)(\dot{a}_2, \dot{b}_2) \in E(A[B])$, in which $a_1 \neq a_2$ and $d((a_1, b_1), (a_2, b_2)) = 2$. Then $d(a_1, a_2) = 2$ and it implies

$$\begin{aligned} (\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}) &\in M_{(a_{1}, b_{1})_{2}(a_{2}, b_{2})}^{A|B|} \Leftrightarrow \qquad (9) \\ d_{A[B]}((\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{1})) &< d_{A[B]}((\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}), (a_{2}, b_{2})) \Leftrightarrow \\ &\min\{d_{A[B]}((\dot{a}_{1}, \dot{b}_{1}), (a_{1}, b_{1})), d_{A[B]}((\dot{a}_{2}, \dot{b}_{2}), (a_{1}, b_{1}))\} < \\ &\min\{d_{A[B]}((\dot{a}_{1}, \dot{b}_{1}), (a_{2}, b_{2})), d_{A[B]}((\dot{a}_{2}, \dot{b}_{2}), (a_{2}, b_{2}))\} \Leftrightarrow \\ & \left\{ \begin{aligned} a_{1} \neq \dot{a}_{1} \Leftrightarrow \dot{a}_{2} \notin \{a_{1}, a_{2}\} \\ a_{1} \neq \dot{a}_{2} \Leftrightarrow \dot{a}_{1} \notin \{a_{1}, a_{2}\} \end{aligned} \right\} \Leftrightarrow \dot{a}_{1}\dot{a}_{2} \in M_{a_{1}\underline{2}}^{A}a_{2}. \end{aligned}$$

Similarly,

$$(\dot{a}_{1}, \dot{b}_{1})(\dot{a}_{2}, \dot{b}_{2}) \in M^{A[B]}_{(a_{2}, b_{2})_{\underline{2}}(a_{1}, b_{1})} \Leftrightarrow \qquad (10)$$

$$\begin{cases} a_{2} \neq \dot{a}_{1} \Leftrightarrow \dot{a}_{1} \notin \{a_{1}, a_{2}\} \\ a_{2} \neq \dot{a}_{2} \Leftrightarrow \dot{a}_{2} \notin \{a_{1}, a_{2}\} \end{cases} \Leftrightarrow \dot{a}_{1} \dot{a}_{2} \in M^{A}_{a_{2}\underline{2}}a_{1}.$$

By (9) and (10) and we know that A[B] is 2-EDB, it implies that A is a 2-EDB graph. For converse, consider that A is 2-EDB and B is locally regular. If $(a_1, b_1), (a_2, b_2) \in V(A[B]), a_1 \neq a_2$ and $((a_1, b_1), (a_2, b_2)) = 2$. Then $d(a_1, a_2) = 2$. Therefore, we have both relations (9) and (10) which are inferred that A[B] is 2-EDB and B is locally regular.

EDB. Thus, A[B] is 2-EDB if and only if A is 2-EDB. Consider that $(a_1, b_1), (a_1, b_2) \in V(A[B]), b_1 \neq b_2$ and $((a_1, b_1), (a_1, b_2)) = 2$. Then $b_1b_2 \notin E(A)$ and

also

$$\begin{aligned} &(\dot{a}_1, \dot{b}_1)(\dot{a}_2, \dot{b}_2) \in M^{A[B]}_{(a_1, b_2)_2(a_1, b_1)} \Leftrightarrow \\ & \begin{cases} \dot{a}_1 = a_1, \ b_2 \dot{b}_1 \in E(B) \ and \ b_1 \dot{b}_1 \notin E(B) \\ \dot{a}_2 = a_1, \ b_2 \dot{b}_2 \in E(B) \ and \ b_1 \dot{b}_2 \notin E(B). \end{aligned}$$

Therefore,

$$\begin{split} M^{A[B]}_{(a_1,b_1)_2(a_1,b_2)} &= \{(a_1, \acute{b}_1)(a_1, \acute{b}_2) | b_1\acute{b}_1, b_1\acute{b}_2 \in E(B), \ b_2\acute{b}_1, b_2\acute{b}_2 \notin E(B)\}.\\ M^{A[B]}_{(a_1,b_2)_2(a_1,b_1)} &= \{(a_1, \acute{b}_1)(a_1, \acute{b}_2) | b_2\acute{b}_1, b_2\acute{b}_2 \in E(B), \ b_1\acute{b}_1, b_1\acute{b}_2 \notin E(B)\}. \end{split}$$

and the fact that *B* is locally regular follows that A[B] is 2-EDB and the proof is completed.

42-EDB property in some subdivisionrelated graphs

In the following, we are going to investigate 2-EDB property in some subdivision-related graphs. Now we need to introduce two subdivision-related graphs which are called R(A) and S(A) and have been introduced by [4, 17].





Figure 1: Graph A



Corollary 41 A graph A is EDB if and only if S(A) is 2-EDB.

Theorem 5 Consider that *A* is a connected graph. Then R(A) is 2-EDB if and only if *A* is a path with |E(A)| = 2.

Proof Suppose that A is a graph, in which |E(A)| = 2. So it is easily seen R(A) is a Friendship graph and consequently 2-EDB. For converse, by assumption contrary we consider that R(A) is 2-EDB, such that A be a graph with |E(A)| > 2. Thus, there is at least a pair of vertices g and h in A with d(g,h) = 2. Suppose that there is a vertex m between g and h. Based on the definition of (A), there is a new vertex x corresponding edge gm in R(A). Then, edge gm is one of the edges of constructed triangle in (A). It is seen that deg(g) = 2. Similarly, there is a new vertex y corresponding edge mh. Then, mh is one of

vertex inserts in each edge of A. In other words, a path of length 2 replaces any edge of A. (See Fig. 1, 2 for an example). R(A) is constructed from A by adding a new

S(A) is constructed from A, such that an extra

vertex corresponding to any edge of A, then connecting any new vertex to the end vertices of the corresponding edge. Replacing any edge of A by a triangle is another way to express R(A). (See Fig. 1,3 for an example).



Figure 3: Graph R(A)

the edges of another triangle in (A). Assume that m and n are another two vertices in A with d(m,n) = 2, such that h is between m and n. In (A), there is a new vertex z corresponding edge hn. Hence, $deg(h) \ge 3$ and $m_{g_2h}^{R(A)} \neq m_{h_2g}^{R(A)}$ and it is contradiction. Thus, it is completed. \Box **Theorem 6** If A be a nontrivial connected

Theorem 6 If A be a nontrivial connected graph with a pendant, then S(A) is not 2-EDB. **Proof** Let $g, h \in V(A)$ with d(g, h) = 2 and u be a pendant vertex. Let m be the new vertex, where g and m and also two vertices h and m are adjacent in (A). Then $m_{g_2h}^{S(A)} = 1$ and $m_{h_2g}^{S(A)} \ge 2$. Therefore, $m_{g_2h}^{S(A)} \neq m_{h_2g}^{S(A)}$, proving the result.

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