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# Qualitative Comparison Between Performance Of Quadrature Rules For Triangles And Squares

**Abdelhamid ZAIDI[1] , Mishael ALHARBI[2]**

[1]*Department of mathematics, College of Science, Qassim University P.O.Box 6644, Buraydah 51452, Saudi Arabia, Orcid: 0000-0003-1305-4959 Email: A.ZAIDI@qu.edu.sa*
[2]*Department of Mathematics, College of Science and Arts in Uglat Asugour, Qassim University, Buraydah, Saudi Arabia., Orcid: 0000-0002-9783-7258 Email: mishael.alharbi@qu.edu.sa*

**Abstract**

Numerical integration on an arbitrary compact set $D \subset \mathbb{R}^2$ generally goes through two phases. The first consists in discretizing D into simple finite elements. The second phase consists in calculating an approximation of the integral on each finite element, to deduce from it an approximation of the integral on D. The triangle and the square are the finite elements most used in affine discretization. Constructing an effective quadrature rule is a laborious and time-consuming task. The quadrature rules discovered so far owe much to the power of supercomputers. This research topic, more than a century old, is still relevant thanks to the continuous growth of supercomputers. We describe in this study the different approaches that allowed the development of positive interior symmetric (PIS) quadrature rules for the triangle and the square. Next, we determine the strength and relative error associated with each rule. Additionally, we use Genz test functions to assess the accuracy of different quadrature rules. Finally, we propose two techniques to reduce the integration error inherent in non-regular integrands.

**Index Terms**

Numerical integration ; Positive symmetric quadrature rules for the triangle ; Positive symmetric quadrature rules for the square ; Numerical integration errors ; Integration error reduction.

## I. INTRODUCTION

The calculation of physical ([1]–[4]), quantum ([5], [6]), chemical ([7], [8]), financial ([9], [10]), economic ([11], [12]), mathematical ([13], [14]), statistical quantities [15], . . . , is often reduced to a calculation of multidimensional integral. Some of these integrals can be computed by analytical methods such as integration by parts, integration by change of variables, etc. However, in the modeling of real phenomena, we are often confronted with integrands, which have no explicit expression. This constraint makes the use of analytical integration methods impossible. Numerical integration methods are a powerful alternative to overcome this kind of difficulty. There are two major classes of numerical methods. The first class includes deterministic methods, while the second includes stochastic methods. Deterministic methods are recommended to approximate integrals over a sub-domain of the affine plane or space, whereas, stochastic methods have proven their effectiveness in approximating integrals over higher-dimensional affine sub-domains.

Let d be an integer in $\{1,2\}$, D a compact subset of $\mathbb{R}^d$, and $\mathbb{L}^1(D)$ the set of integrable real-valued functions on D. Let $w(\cdot)$ be a strictly positive function on the open set of D. For all $f \in \mathbb{L}^1(D)$, we denote by I[f] the weighted integral of f over D:

$$I[f] = \int_D w(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}, \qquad \text{where } \mathbf{x} = (x_1, x_2) \text{ if } d = 2. \quad (1)$$

We call a quadrature method any formula which approximates the I [f] by a weighted sum of f on a set of points sampled on the integration domain D. More precisely, for all n ∈ N, a n + 1-node quadrature rule, denoted $Q_{n+1}$, is characterized by the given of n+1 points $(\mathbf{x}_{i,n+1})_{0\le i\le n}$, and n + 1 real constants $(\mathbf{w}_{i,n+1})_{0\le i\le n}$, such that:

$$I[f] \approx Q_{n+1}[f] = \sum_{i=0}^{n} w_{i,n+1} f(x_{i,n+1}) \qquad (2)$$

and,

$$I[f] = Q_{n+1}[f] \qquad \text{if} \qquad f \in \mathbb{P}(D), \qquad (3)$$

where $\mathbb{P}(D) \subset \mathbb{L}^1(D)$ is the largest possible subset of $\mathbb{L}^1(D)$ for which the quadrature rule (2) is exact. In most cases, $\mathbb{P}(D)$ is taken equal to the vector space $\mathbb{P}_{d,m}$ of polynomials in d variables of degree less than or equal to a given integer m. More precisely,

if $d = 1$, then $\mathbb{P}_{1,m} = \mathbb{R}[X]$,

$$\text{if} \quad d = 2, \quad \text{then} \quad \mathbb{P}_{2,m} = \mathbb{R}[X, Y].$$
(4)

Closed intervals are the unique compact sets of the affine line. However, compacts of the affine plane can have various geometric shapes. The construction of a quadrature rules for a compact D of the affine plane is much more laborious than than that for other geometric shapes, such as circles, squares, or polygons with more than 4 sides. This is due to the fundamental role played by the triangle in the discretization of compact surfaces using triangulation techniques. The construction of quadrature rules for the triangle and the square has been going on for more than a century and is still relevant thanks to the continued growth of supercomputers. We believe that this subject will always be relevant, as long as the rules of quadrature have not reached an infinite strength as in the case of the segment.

In the following, unless otherwise stated, the symbol D will designate either the interval $\mathbb{I}_0 = [-1,1]$, or the triangle $\mathbb{T}_0 = [(0,0);(1,0);(0,1)]$, or the square $\mathbb{S}_0 = [-1,1]^2$.

If the domain of integration is arbitrary, it is possible to approximate it by a finite union of rectangles or triangles,

$$D = \cup \left( \bigcup_{j \in J} D_j \right) \cup A. \qquad (5)$$

There are two types of errors inherent in the implementation of a numerical deterministic method. The first error, which we will call mesh error, and which we will denote by $E^{(m)}$,

Therefore, for a given quadrature method, we approximate integral I[f] by the following weighted sum:

$$I[f] \approx \sum_{j \in J} \sum_{i=1}^{n_j} w_{i,n_j} f\left(\mathbf{x}_{i,n_j}\right) + E_j^{(q)} + E^{(m)}, \qquad (9)$$

and the total error which results from this approximation is

$$E = E^{(q)} + E^{(m)}. \qquad (10)$$

To reduce the total error, it is necessary to simultaneously reduce the mesh and quadratic errors. More precisely, to reduce

1) the error $E^{(m)}$, it suffices to refine the mesh of the integration domain D,
2) to reduce the error $E^{(q)}$, it suffices to increase the number $n_j$, of points sampled in $D_j$.

## II. MATHEMATICAL FORMULATION OF THE QUADRATURE PROBLEM

Notice that:

$$P(x) = \sum_{i=0}^{m} a_i x^i, \qquad (11)$$

that for the segment. Indeed, the complexity of constructing a quadrature rule depends on the geometric shape of the integration domain D (triangle [16], square [17], circle [18], arbitrary polygons [19]), and especially on the extent of the set $\mathbb{P}_{2,m}$, characterized by the parameter m. Research on quadrature rules for the triangle is more flourishing

results from the approximation of the integration domain D, by a finite union of elementary domains $\left(D_j\right)_{j \in J}$ of the same nature (generally either rectangles or triangles):

$$\iint_D f(x)dx = \sum_{j \in J} \iint_{D_j} f(x)dx + E^{(m)} \qquad (6)$$

where

$$E^{(m)} = \iint_A f(x)dx. \qquad (7)$$

The second error, which we will call quadrature error, and which we will denote by $E^{(q)}$ comes from the approximation of the integral over each elementary domain $\left(D_j\right)_{j \in J}$ by a weighted sum of the integrand on a set of points $\left\{x_{1,n_j}, \dots, x_{n_j,n_j}\right\}$ sampled on the elementary domain:

$$\begin{aligned} \iint_{D_j} f(\mathbf{x})d\mathbf{x} &= \sum_{i=1}^{n_j} w_{i,n_j} f\left(\mathbf{x}_{i,n_j}\right) + E_j^{(q)} \\ E^{(q)} &= \sum_{j \in J} E_j^{(q)} \end{aligned} \qquad (8)$$

A. Study objectives

This study constitutes an in-depth comparative analysis of the most effective PIS quadrature rules for triangles and squares. Additionally, it provides the reader with the tools to understand and use these rules. More precisely,

1) we describe here the different approaches that allowed the development of PIS quadrature rules for triangles and squares,
2) we determine the strength and relative error associated with each rule,
3) we use Genz test functions to assess the accuracy of different quadrature rules,
4) we propose two techniques to reduce the integration error inherent in non-regular integrands.

1) if $P \in \mathbb{P}_{1,m}$, then, there exist $m+1$ unique reals coefficients $(a_i)_{0 \le i \le m}$ such that:

which means that $\mathbb{P}_{1,m}$ is of dimension:

$$dim(\mathbb{P}_{1,m}) = m + 1 = m_1. \qquad (12)$$

Moreover, $B_{1,m} = \{ x^i \mid i = 0, \ldots, m \}$ is the canonical basis of $\mathbb{P}_{1,m}$.

2 if $P \in \mathbb{P}_{2,m}$, then, there exist $(m+1)(m+2)/2$ unique reals coefficients $(a_{i,j})_{0 \le i+j \le m}$ such that:

$$P(x,y) = \sum_{0 \le i+j \le m} a_{i,j} x^i y^j, \qquad (13)$$

which means that $\mathbb{P}_{2,m}$ is of dimension:

$$dim(\mathbb{P}_{2,m}) = \frac{(m+1)(m+2)}{2} = m_2. \qquad (14)$$

Therefore, the nodes $(\boldsymbol{x}_{i,n+1})_{0 \le i \le n}$, and the weights $(w_{i,n+1})_{0 \le i \le n}$ satisfy the system $A\boldsymbol{w} = \boldsymbol{b}$, where

$$,A = \begin{pmatrix} q_1(\boldsymbol{x}_{0,n+1}) & \cdots & q_1(\boldsymbol{x}_{n,n+1}) \\ q_2(\boldsymbol{x}_{0,n+1}) & \cdots & q_2(\boldsymbol{x}_{n,n+1}) \\ \vdots & \cdots & \vdots \\ q_{m_d}(\boldsymbol{x}_{0,n+1}) & \cdots & q_{m_d}(\boldsymbol{x}_{n,n+1}) \end{pmatrix}; \quad \boldsymbol{w} = \begin{pmatrix} w_{0,n+1} \\ w_{1,n+1} \\ \vdots \\ w_{n,n+1} \end{pmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} \langle 1, q_1 \rangle_w \\ \langle 1, q_2 \rangle_w \\ \vdots \\ \langle 1, q_{m_d} \rangle_w \end{pmatrix}. \qquad (17)$$

Moreover, $\mathcal{B}_{2,m} = \{ x^i y^j \mid i + j = 0, \ldots, m \}$ is the canonical basis of $\mathbb{P}_{2,m}$. Recall that for all $q_k \in \mathcal{B}_{d,m}, k = 1, \ldots, m_d$, we have:

$$Q_{n+1}[q_k] = \int_D w(\boldsymbol{x}) 1 \cdot q_k(\boldsymbol{x}) d\boldsymbol{x} = \langle 1, q_k \rangle_w = \sum_{i=0}^{n} w_{i,n+1} q_k(\boldsymbol{x}_{i,n+1}) \qquad (15)$$

where $\langle . , . \rangle_w$ is the inner product on $\mathbb{P}_{d,m_d}$, defined for all $p, q \in \mathbb{P}_{d,m_d}$ by:

$$\langle p, q \rangle_w = \int_D w(\boldsymbol{x}) p(\boldsymbol{x}) q(\boldsymbol{x}) d\boldsymbol{x} \qquad (16)$$

According to ([20], [21]), the numerical resolution of system (17) is difficult because it is ill-conditioned. To overcome this drawback, it suffices to replace $\boldsymbol{\mathcal{B}}_{d,md}$ by an echelon-degree orthogonal basis,

$$\boldsymbol{\mathcal{B}}^*_{d,m} = \{ \tilde{q}_0, \ldots, \tilde{q}_{m_d} \mid \tilde{q}_0 \text{ is constant} \}. \qquad (18)$$

with respect to the inner product (16). Therefore, the nodes and the weights of the quadrature rule $Q_{n+1}$, satisfy the system $\tilde{A}\boldsymbol{w} = \tilde{\boldsymbol{b}}$, where,

$$\tilde{A} = \begin{pmatrix} \tilde{q}_1(\boldsymbol{x}_{0,n+1}) & \cdots & \tilde{q}_1(\boldsymbol{x}_{n,n+1}) \\ \tilde{q}_2(\boldsymbol{x}_{0,n+1}) & \cdots & \tilde{q}_2(\boldsymbol{x}_{n,n+1}) \\ \vdots & \cdots & \vdots \\ \tilde{q}_{md}(\boldsymbol{x}_{0,n+1}) & \cdots & \tilde{q}_{md}(\boldsymbol{x}_{n,n+1}) \end{pmatrix};$$

$$\boldsymbol{w} = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix}; \quad \tilde{\boldsymbol{b}} = \begin{pmatrix} \langle 1, \tilde{q}_0 \rangle_w \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (19)$$

A. Examples of orthogonal bases

Here we give orthogonal bases inside the integration reference domains $\mathbb{I}_0, \mathbb{T}_0$ and $S_0$ defined above. Using these bases and appropriate affine transformations we can deduce orthogonal bases for any interval, triangle or square.

1 $D = \mathbb{I}_0$ : Let $w(x) = (1-x)^\alpha (1+x)^\beta$, where $\alpha, \beta > -1$, then $(q_n)_{n \ge 0}$ coincide with Jacobi polynomials, denoted by $\left(P_n^{[\alpha,\beta]}\right)_{n \ge 0}$. Moreover, $\mathcal{B}^*_{1,m} = \left\{ P_i^{[\alpha,\beta]} \mid i = 0, \ldots, m \right\}$ is an orthogonal basis of $\mathbb{P}_{1,m}$ with respect to the inner product:

$$\langle p, q \rangle_{\boldsymbol{w}} = \int_{\mathbb{I}_0} w(\boldsymbol{x}) p(\boldsymbol{x}) q(\boldsymbol{x}) d\boldsymbol{x}. \qquad (20)$$

2 $D = \mathbb{S}_0$ : Let $w(x,y) = (1-x)^\alpha (1+y)^\beta$, where $\alpha, \beta > -1$. Let $\Psi_{i,j} = P_t^{[\alpha,0]}(x) P_j^{[0,\beta]}(y)$, for all $i, j \in \mathbb{N}$. Then, $\boldsymbol{\mathcal{B}}^*_{2,m} = \{\Psi_{i,j} \mid i + j = 0, \ldots, m_2\}$ is an orthogonal basis of $\mathbb{P}_{2,m}$ with respect to the inner product:

$$\langle p, q \rangle_w = \iint_{\mathbb{S}_0} w(\boldsymbol{x}) p(\boldsymbol{x}) q(\boldsymbol{x}) d\boldsymbol{x}. \qquad (21)$$

3 $D = \mathbb{T}_0$ : Let $\mathbb{T} = [(-1, -1/\sqrt{3}); (0, 2/\sqrt{3}); (1, -1/\sqrt{3})]$, $w(x,y) = 1$. Using the orthogonal basis $\{\overline{K}_{i,j}; 0 \le i + j \le m\}$ with respect to the inner product:

$$\langle p, q \rangle_{\boldsymbol{w}} = \iint_{\mathbb{T}} w(x,y) p(\boldsymbol{x}) q(\boldsymbol{x}) d\boldsymbol{x}. \qquad (22)$$

given in [21], and the change of variables rule, it is easy to deduce that $\boldsymbol{\mathcal{B}}^*_{2,m} = \{ \Upsilon_{i,j} \mid 0 \le i + j \le m \}$ is an orthogonal basis of $\mathbb{P}_{2,m}$ with respect to the inner product:

$$\langle p, q \rangle_w = \iint_{\mathbb{T}_0} w(x,y) p(\boldsymbol{x}) q(\boldsymbol{x}) d\boldsymbol{x}, \qquad (23)$$

where,

$$\Upsilon_{i,j}(x,y) = (1-y)^i P_i^{[0,0]} \left( \frac{2x + y + 1}{1 - y} \right) P_j^{[2i+1,0]}(2y - 1). \qquad (24)$$

B. Transformation of system (17) into a least squares problem

Let $x_{[0:n]} = (\boldsymbol{x}_{0,n+1}, \ldots, \boldsymbol{x}_{n,n+1})$, then we can write system (17) as $F(\boldsymbol{x}_{[0:n]},\ \boldsymbol{w}) = 0$, where,

$$F(\boldsymbol{x}_{[0:n]},\ \boldsymbol{w}) = A\boldsymbol{w} - \boldsymbol{b}. \qquad (25)$$

In many situations, it is more convenient to turn the nonlinear system (17) into a least squares problem, especially when the Jacobian of $F$ is not of full rank. In this case, the numerical solution could well be determined by the Levenberg-Marquardt algorithm [22], which is considered a regularization of the Gauss-Newton one.

Here we show how to transform the nonlinear system (17) into a least squares problem, in the case where $D = \mathbb{S}_0$ or $D = \mathbb{T}_0$. To do this, we use the results stated in Lemmas 2.1 and 2.2.

Lemma 2.1: For all $i, j \in \mathbb{N}$, we have:

$$\iint_{T_0} x^i y^j dxdy = \frac{i!\,j!}{(i+j+2)!} \qquad (26)$$

Lemma 2.2: For all $i, j \in \mathbb{N}$, we have:

$$\iint_{\mathcal{S}_0} x^i y^j = \begin{cases} 0 & \text{if } i \text{ or } j \text{ is odd,} \\ \dfrac{4}{(i+1)(j+1)} & \text{if } i \text{ and } j \text{ are even,} \end{cases} \qquad (27)$$

Let $q_{i,j}(x, y) = x^i y^j$ be the $(i, j)$-element of the canonical basis of $\mathbb{P}_{2,m}$. Knowing that $Q_{n+1}[q_{i,j}] = I[q_{j,1}]$, and by virtue of the symmetry of the quadrature rules, it then sufficient to retain from system (17) only the equations corresponding to $\{0 \le i + j \le m$ and $j \le i\}$. The least square criterion corresponding to each integration domain can be written as:

$$\mathcal{C}(\boldsymbol{x}_{[0:n]}, \boldsymbol{w}) = \sum_{i=0}^{n} \sum_{j=0}^{1} \left(Q_{n+1}[q_{i,j}] - I[q_{i,j}]\right)^2. \quad (28)$$

The nodes $(\boldsymbol{x}_{i,n+1})0 \le 1 \le n$ and the weights $(w_{i,n+1})0 \le 1 \le n$ are obtained by minimizing the criterion $\mathcal{C}(\cdot,\cdot)$ with respect to the arguments $(x_{[0:n]}, w)$.

C. Properties of a quadrature rule

Let $k \in \{2,3,4\}$, and $A_1, \ldots, A_k$ be the vertices of the polygon $D = [A_1; \ldots; A_k]$ :

- if $k = 2$, then $D = \mathbb{I}_0$,
- if $k = 3$, then $D = \mathbb{T}_0$,
- if $k = 4$, then $D = \mathbb{S}_0$.

Definition 2.1 (Barycentric coordinates): Let $M$ be a point inside $D = [A_1; \ldots; K_k]$, whose coordinates are $(x, y)$ in an affine frame $(O, \mathbf{1}, \boldsymbol{j})$. Then, there exists a unique vector $\alpha = (\alpha_1, \ldots, \alpha_k)^t \in [0,1]^k$ such that:

$$\overrightarrow{MA_1} + \cdots + \vec{M}\vec{A}_k = \vec{0} \text{ and } \alpha_1 + \cdots + \alpha_k = 1. \qquad (29)$$

The vector $\boldsymbol{\alpha}$ defines the barycentric coordinates of $M$.

In order not to favor one vertex over the others, it is desirable that the quadrature rule be symmetric in the sense given by Definition 2.2 :

Definition 2.2 (symmeric quadrature rule): A $n + 1$-node quadratic rule is said to be symmetric, if the nodes $\boldsymbol{x}_{0,n+1}$, . . . , $\boldsymbol{x}_{n,n+1}$ are invariant under all affine symmetries which leave the polygon D invariant.

Definition 2.3 (orbit of a point): Let $M$ be a point inside the polygon $D = [A_1, \ldots, A_k]$ whose barycentric coordinates are $(\alpha_1, \ldots, \alpha_k)$. The orbit generated by $M$, denoted by $\mathcal{O}(M)$, consists of all the points whose barycentric coordinates are permutations of the $k$-tuple $(\alpha_1, \ldots, \alpha_k)$. In a barycentric coordinate system, any symmetry which leaves the polygon $D$ invariant is defined as a permutation of the barycentric coordinates. Therefore, using Definitions 2.2 and 2.3, one can prove [21] that:

Corollary 2.1 (symmerric quadrature rule): A $n + 1$-node quadrature rule is said to be symmetric if:

1) the set of nodes is a disjoint union of orbits,
2) for a given orbit, the weights of its nodes are identical.

A second desirable property for a $n + 1$-node quadrature rule is to be of maximum strength in the sense given by Definition 2.4

Definition 2.4 (strength of quadrature rule): The symmetric $n + 1$-node quadrature rule $Q_{n+1}$ is said to be of strength $m$, if it satisfies the following condition:

$$\begin{array}{l} Q_{n+1}[f] = I[f],\ \forall f \in \mathbb{P}_{d,m} \\ \exists f \in \mathbb{P}_{d,m+1} \ \mid\ Q_{n+1}[f] \neq I[f]. \end{array} \qquad (30)$$

A third desirable property for a $n + 1$-node quadrature rule is to be positive and interior in the sense given by Definition 2.5:

Definition 2.5 (PI quadrature rule): A quadrature rule whose nodes $\boldsymbol{x}_{0,n+1}, \ldots, \boldsymbol{x}_{n,n+1}$, and weights $w_{0,n+1}, \ldots, w_{n,n+1}$ is said to be positive and interior, and denoted PI, if

1 $x_{i,n+1} \in D$, for all $i = 0, \ldots, n$,

2 $w_{i,n+1} > 0$, for all $i = 0, \ldots, n$.

In the following, a quadrature rule satisfying the properties given in Definitions 2.2 and 2.5 will be denoted PIS. Quadrature rules for the polygon $D$ can be asymmetric, non-positive, exterior, or PIS.

Finally we define the notions of efficiency and optimality for a given quadrature rule of strength $m$, to make the comparison between all the quadrature rules of the same strength.

Definition 2.6 (efficient quadrature rule): For a given strength $m$, a quadrature rule is said to be efficient if it requires the fewest nodes among all the rules of strength $m$ known so far.

Definition 2.7 (optimal quadrature rule): For a given strength $m$, a quadrature rule is said to be optimal if the number of nodes it uses is equal to the lower bound of the number of nodes necessary to construct any quadrature rule of strength $m$.

Remark 2.1: For squares or triangles, an optimal quadrature rule of strength $m$ requires $n^* + 1$ nodes (see[17] and [23]), where:

$$n^* = \frac{(m+2)(m+4)}{8} - 1, \text{ if } m \text{ is even},$$
$$n^* = \frac{(m+1)(m+3)}{8} + \left\lfloor\frac{d+1}{4}\right\rfloor - 1, \text{ if } m \text{ is odd}. \quad (31)$$

## III. QUADRATURE RULES FOR THE INTERVAL $\mathbb{I}_0$

For univariate integrals, we have many results for the system (19), namely:

1) for all $n \in \mathbb{N}$, the polynomial $\tilde{q}_{n+1}$ has $n+1$ real simple roots,
2) the nodes $\boldsymbol{x}_{0,n+1}, . . . , \boldsymbol{x}_{n,n+1}$ are the roots of $\tilde{q}_{n+1}$,
3) the matrix $\tilde{A}$ is invertible,
4) the weights $w_{0,n+1}, \dots, w_{n,n+1}$ are the unique solution of the linear system (19),
5) for all $i = 0, . . . , n, w_{i,n+1} > 0$,
6) $I = Q_{n+1}[f]$, for all $f \in \mathbb{P}_{1,2n+1}$.
7) $I \neq Q_{n+1}[f]$, for all $f \in \mathbb{P}_{1,2n+2}$.

moreover,

1) if $w(x) = 1$, then $(q_n)_{n\geq 0}$ coincide with Legendre polynomials ([24], p. 23),
2) if $w(x) = 1 - x^2$, then $(q_n)_{n\geq 0}$ coincide with Lobatto polynomials,
3) if $w(x) = 1/\sqrt{1 - x^2}$, then $(q_n)_{n\geq 0}$ coincide with Chebyshev polynomials. Besides, for all $i = 0, \dots, n$, we have ([24], pp. 28-30):

$$\boldsymbol{x}_{i,n+1} = cos\left(\frac{2i+1}{2n+2}\pi\right),$$
$$w_{i,n+1} = \frac{\pi}{n+1}. \quad (32)$$

A. Accuracy of quadrature Rules for $\mathbb{I}_0$

Denote by $\mathcal{C}^{n+1}(\mathbb{I}_0)$ the set of $n+1$-times continuously differentiable functions on the set $\mathbb{I}_n$.

Theorem 3.l: If $f \in \mathcal{C}^{n+1}(\mathbb{I}_0)$, then there exists a positive constant $C$, such that:

$$|Q_{n+1}[f] - I[f]| \leq \frac{C}{2^n(n+1)!} \quad (33)$$

**Proof**: Denote by $P_{n+1}f$ the interpolation polynomial of the function $f$ at nodes $\boldsymbol{x}_{0,n}, \dots, \boldsymbol{x}_{n,n+1}$. If $f \in \mathcal{C}^{n+1}([-1,1])$, then ([25], p. 878):

$$\forall t \in [-1,1], \exists \xi_t \in [-1,1] \mid f(t) - P_{n+1}f(t) = \frac{f^{(n+1)}(\xi_t)}{(n+1)!}\prod_{i=0}^{n}\left(t - \boldsymbol{x}_{i,n+1}\right). \quad (34)$$

Therefore,

$$\int_{-1}^{1} f(t)w(t)dt - \int_{-1}^{1} P_{n+1}f(t)w(t)dt = \int_{-1}^{1}\frac{f^{(n+1)}(\xi_t)}{(n+1)!}w(t)\prod_{t=0}^{n}\left(t - \boldsymbol{x}_{i,n+1}\right)dt \quad (35)$$

Denote by $E_{n+1}$ the difference between $Q_{n+1}[f]$ and $I[f]$. Then, we have:

$$\begin{aligned} E_{n+1} &= I[f] - I[P_{n+1}f] \\ &= \int_{\mathbb{I}_0} f(t)w(t)dt - \int_{\mathbb{I}_0} P_{n+1}f(t)w(t)dt \\ &= \int_{\mathbb{I}_0} w(t)f(t)dt - \sum_{i=0}^{n} w_{i,n+1}P_{n+1}f\left(\boldsymbol{x}_{i,n+1}\right). \end{aligned}$$
(36)

Let $M_{n+1}, N$, and $\Omega$ be the following constant reals:

$$\begin{aligned} M_{n+1} &= \text{m } \left\{\left|f^{(n+1)}(t)\right|; t \in [-1,1]\right\}, \\ N &= \text{m } \left\{\left|\prod_{i=0}^{n} \left(t - \boldsymbol{x}_{i,n+1}\right)\right|; t \in [-1,1]\right\}, \\ \Omega &= \int_{\mathbb{I}_0} w(t)dt. \end{aligned}$$
(37)

According to ([25], p. 889), we have:

$$N = \frac{1}{2^n}. \quad (38)$$

Let $C = M_{n+1}\Omega$. Then, using relations (35), and (37) and (38), we get:

$$|E_{n+1}| \leq \frac{C}{2^n(n+1)!}.$$

Inequality (33) show that if the integrand $f$ is sufficiently regular over $I_0$, then $Q_{n+1}[f]$ converges to $I[f]$ with much faster speed than Trapezoidal or Simpson's methods ([25], p. 885-886).

Remark 3.l: Note that the upper bound given by relation (33) was calculated under the assumption $f \in \mathcal{C}^{n+1}(\mathbb{I}_0)$. . In practice, if $f$ were of class $\mathcal{C}^{\kappa+1}(\mathbb{I}_0)$, with $\kappa < n$, then the mor $E_{n+1}$ would be bounded from above by $C'/(2^\kappa(\kappa+1)!)$, where $C' = M_{\kappa+1}\Omega$.

## IV. QUADRATURE RULES FOR THE SQUARE $\mathbb{S}_0$

Let $(O, \boldsymbol{i}, j)$ be the canonical basis of the affine plane, where $O$ is the the centroid of $\mathbb{S}_0$. For II $M \in \mathbb{S}_0$, there exists unique real vector $(x, y) \in [-1,1]^2$ such that:

$$\overrightarrow{OM} = xi + y\boldsymbol{j}. \qquad (39)$$

The vector $x = (x, y)^t$ defines the affine coordinates of $M$ in the affine basis $(O, i, j)$. We seek characterize the orbits introduced in Definition 2.3. Note that there are eight symmetries that leaves the square $\mathbb{S}_0$ invariant:

1) the identity map $Id_2$,
2) the reflections across lines $(A, C), (B, D), (O, x)$ and $(O, y)$,
3) the rotations with center $O$ and angles $\frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$.

A point $M = (x, y)^t$ in the square $\mathbb{S}_0$, can have one of the following affine coordinates:

type-1: $M = (0,0)$,
type-2: $M = (x, 0)$, where $x \neq 0$,
type-3: $M = (x, x)$, where $x \neq 0$, (40)
type-4: $M = (x, y)$, where $x \neq y \neq 0$

Therefore, $\mathcal{O}(M)$ consists of all the points whose affine coordinates are of the form $(\pm x, \pm y), \pm y, \pm x)$. Thus,

1) if $M$ is of type-1, then $card\ (\mathcal{O}(M)) = 1$,
2) if $M$ is of type-2, then $card\ (\mathcal{O}(M)) = 4$,
3) if $M$ is of type-3, then $card\ (\mathcal{O}(M)) = 4$,
4) if $M$ is of type-4, then $card\ (\mathcal{O}(M)) = 8$,

Consequently, a $n+1$-node quadrature rule consists of $n_1$ type-1 orbits, $n_2$ type-2 orbits, $n_3$ ree-3 and $n_4$ type-4 orbits such that:

$$n + 1 = n_1 + 4n_2 + 4n_3 + 8n_4. \qquad (41)$$

Since there is a unique type-1 orbit, then $n_1 \in \{0,1\}$, and:

$n + 1 = 4n_2 + 4n_3 + 8n_4$ or $n = 4n_2 + 4n_3 + 8n_4$. (42)

Knowing that,

1) for a given orbit, the weights of its nodes are identical [21],
2) the elements of a given type-2 orbit require one parameter $x$,
3) the elements of a given type-3 orbit require one parameter $x$,
4) the elements of a given type-4 orbit require two parameters $(x, y)$,

A symmetric $n+1$-node quadrature rule for the square requires $n_1 + 2n_2 + 2n3 + +3n4$ parameters:

1) $n_1 + n_2 + n_3 + n_4$ parameters to define the weights,
2) $n_2 + n_3 + 2n_4$ parameters to define the coordinates of the nodes.

Thus, to determine a $n+1$-node symmetric quadrature rule, we must first decompose $n+1$ into a quadruplet $(n_1, n_2, n_3, n_4)$, then rewrite system (17) or criterion (28),taking into account the conditions mentioned in Corollary 2.1.

### A. Quadrature rules for arbitrary square

Let $A_0 = (-1,-1), B_0 = (1,-1), C_0 = (1,1)$, and $D_0 = (-1,1)$ be the vertices of the square $\mathbb{S}_0 = [A_0; B_0; C_0; D_0]$. Let $\mathbb{S}'$ be an arbitrary square $[A'; B'; C'; D']$ such that:

$$A' = (a_1, a_2)^t\ ;\ B' = (b_1, b_2)^t\ ;\ C' = (c_1, c_2)^t\ ;\ D' = (d_1, d_2)^t. \qquad (43)$$

Let $F$ be the affine map, which transforms $\mathbb{S}_0$ into $\mathbb{S}'$. Then,

$$F(A_0) = A'\ ;\quad F(B_0) = B'\ ;\quad F(C_0) = C'\ ;\quad F(D_0) = D'. \qquad (44)$$

Let $M = (x, y)^t \in \mathbb{S}_0$, and $M' = (x', y')^t = F(M)$. Then,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} b_1 - a_1 & d_1 - a_1 \\ b_2 - a_2 & d_2 - a_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} b_1 + d_1 \\ b_2 + d_2 \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} b_1 + d_1 \\ b_2 + d_2 \end{pmatrix}$$

where,

$$P = \frac{1}{2}\begin{pmatrix} b_1 - a_1 & d_1 - a_1 \\ b_2 - a_2 & d_2 - a_2 \end{pmatrix} \qquad (45)$$

Since,

$$|\ det\ (P)| = \frac{1}{4} Area(\mathbb{S}'), \qquad (46)$$

then, by using the change of variables formula, we get:

$$\iint_{\mathbb{S}'} f(x',y')dx'dy' = \frac{Area\ (\mathbb{S}')}{4}\iint_{\mathbb{S}_0} f(F(x,y)^t)dxdy. \qquad (47)$$

By approximating $I[f \circ F]$ by $Q_{n+1}[f \circ F]$, we get,

$$\begin{aligned}\iint_{\mathbb{S}'} f(x',y')dx'dy' &\simeq \frac{Area\ (\mathbb{S}')}{4}\sum_{i=0}^{n} w_{i,n+1} f\left(F\left(x_{i,n+1}, y_{i,n+1}\right)\right), \\ &= \sum_{i=0}^{n} w'_{i,n+1} f\left(x'_{i,n+1}, y'_{i,n+1}\right),\end{aligned} \qquad (48)$$

where, for all $i = 0, \ . \ . \ ., n$,

$$w'_{i,n+1} = \frac{Area\ (\mathbb{S}')}{4} w_{i,n+1} \text{ and } \left(x'_{i,n+1}, y'_{i,n+1}\right)^t = F\left(\left(x_{i,n+1}, y_{i,n+1}\right)^t\right). \qquad (49)$$

In the following, to avoid overloading the notations, we write:

$$Q_{n+1}[f] = \sum_{i=0}^{n} w_{i,n+1} f(\boldsymbol{x}_{i,n+1}), \qquad (50)$$

to represent the $n+1$-node quadrature rule on any square $\mathbb{S}'$ keeping in mind the relation (49).

B. Accuracy of the quadrature rule for the square

Theorem 4.1: Let $\mathbb{S}$ be a square whose edge equals $h$. Let $E_{n+1}[f]$ be the error made in approximating $I[f]$ by $Q_{n+1}[f]$ :

$$E_{n+1}[f] = \iint_D w(x,y)f(x,y)dxdy - \sum_{i=0}^{n} w_{i,n+1} f\left(x_{i,n+1}, y_{i,n+1}\right) \qquad (51)$$

If $f \in \mathcal{C}^{m+1}(\mathbb{S})$, then there exists a positive constant $M_{m+1}$ such that:

$$|E_{n+1}[f]| \leq M_{m+1} \frac{2^{m+2}}{(m+1)!}(h)^{m+3}, \qquad (52)$$

where $m$ is the strength of $Q_{n+1}$.

**Proof :** Let $(x_0, y_0)$ be a point inside the open rectangle $\mathbb{S}$. The Taylor-Lagrange expansion of $f$ at $(x_0, y_0)$ ([26], p. 59), ensures the existence of a point $(\lambda, \theta) \in \mathbb{S}$, a polynomial $P_m \in \mathbb{P}_{2,m}$, and a polynomial $R_m \in \mathbb{P}_{2,m+1}$, such that:

$$f(x,y) = P_m(x,y) + R_m(x,y), \qquad (53)$$

$$\begin{aligned} R_m(x,y) &= \frac{1}{(m+1)!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{m+1} f(\lambda,\theta) \\ &= \frac{1}{(m+1)!}\left[\sum_{i=1}^{m+1}\binom{m+1}{i}(x-x_0)^{m+1-i}(y-y_0)^i \frac{\partial^{m+1} f(\lambda,\theta)}{\partial x^{m+1-i}\partial y^i}\right] \\ &= \frac{1}{(m+1)!}\left[(x-x_0)^{m+1}\frac{\partial^{m+1} f(\lambda,\theta)}{\partial x^{m+1}} + \binom{m+1}{1}(x-x_0)^m (y-y_0)^1 \frac{\partial^{m+1} f(\lambda,\theta)}{\partial x^m \partial y^1} + \cdots\right. \\ &\left. + \binom{m+1}{i}(x-x_0)^{m+1-i}(y-y_0)^i \frac{\partial^{m+1} f(\lambda,\theta)}{\partial x^{m+1-i}\partial y^i} + (y-y_0)^{m+1}\frac{\partial^{m+1} f(\lambda,\theta)}{\partial y^{m+1}}\right] \end{aligned} \qquad (54)$$

is the Lagrange form of the remainder. Since $P_m \in \mathbb{P}_{2,m}$, then,

$$\begin{aligned} E_{n+1}[f] &= E_{n+1}[P_m] + E_{n+1}[R_m] \\ &= \iint_{\mathbb{S}} P_m(x,y)dxdy + \iint_{\mathbb{S}} R_m(x,y)dxdy - Q_{n+1}[P_m] - Q_{n+1}[R_m], \\ &= \frac{1}{(m+1)!}\left[\iint_{\mathbb{S}} (x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{m+1} f(\lambda,\theta)dxdy - Q_{n+1}[R_m], \end{aligned}$$

(55)

where,

$$Q_{n+1}[R_m] = \sum_{i=1}^{n} w_{i,n+1} \iint_{\mathbb{S}} \left[\left(x_{i,n+1} - x_0\right)\frac{\partial}{\partial x} + \left(y_{i,n+1} - y_0\right)\frac{\partial}{\partial y}\right]^{m+1} f(\lambda_i, \theta_i) dxdy \quad (56)$$

Since $(\lambda, \theta)$ depends on $(x, y)$, then $(\lambda_i, \theta_i)$ depends also on $\left(x_{i,n+1}, y_{i,n+1}\right)$. Let $M_{m+1}$ be the upper bound of all the $(m+1)$ derivatives of $f$ over $\mathbb{S}$ :

$$M_{m+1} = \max_{(x,y)\in\mathbb{S}} \left|\frac{\partial^{m+1} f(x,y)}{\partial x^{m+1-i}\partial y^i}\right|. \quad (57)$$

On the other hand, we have:

$$\begin{aligned}\left|\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]^{m+1} f(x,y)\right| &\leq M_{m+1}\left[\sum_{i=1}^{m+1} \binom{m+1}{i} |x - x_0|^{m+1-i}|y - y_0|^i\right] \\ &\leq M_{m+1}(|x - x_0| + |y - y_0|)^{m+1} \\ &\leq M_{m+1}(2h)^{m+1}.\end{aligned}$$

(58)

Since $1 \in \mathbb{P}_{2,m+1}$, then,

$$\iint_{\mathbb{S}} 1dxdy = Area(\mathbb{S}) = h^2 = \sum_{i=0}^{n} w_{i,n} \quad (59)$$

We deduce from (57), (58), and (59) that,

$$\begin{aligned}|E_{n+1}| &\leq \frac{M_{m+1}}{(m+1)!}\left[\iint_{\mathbb{S}} (2h)^{m+1}dxdy + \sum_{i=0}^{n} w_{i,n}(2h)^{m+1}\right] \\ &\leq \frac{M_{m+1}}{(m+1)!}(2h)^{m+1}\left[Area(\mathbb{S}) + \sum_{i=0}^{n} w_{i,n}\right] \quad (60) \\ &\leq \frac{M_{m+1}}{(m+1)!}(2h)^{m+1}[2Area\,(\mathbb{S})] \\ &= M_{m+1}\frac{2^{m+2}}{(m+1)!}(h)^{m+3}\end{aligned}$$

On the other hand, we know that,

$$\sum_{m=0}^{+\infty} \frac{2^m}{m!} = exp\,(2)$$

Since $m$ goes to $+\infty$, when $n$ goes to $+\infty$, then

$$\lim_{n\to+\infty} |E_{n+1}| = \lim_{m\to+\infty} \frac{2^m}{m!} = 0.$$

Relation (52) show that if the integrand $f$ is sufficiently regular over the square $\mathbb{S}$, then $Q_{n+1}[f]$ converges quickly to $I[f]$.

Remark 4.1: Note that the upper bound given by relation (52) was calculated under the assumption $f \in \mathcal{C}^{m+1}(\mathbb{S})$. In practice, if $f$ were of class $\mathcal{C}^{\kappa+1}(\mathbb{S})$, with $\kappa < m$, then the error $E_{n+1}$ would be bounded from above by $M_{\kappa+1}2^{\kappa+2}h^{\kappa+3}/(\kappa+1)!$.

C. Tensor-product quadrature for the square $\mathbb{S}_0$

Having one-dimensional Gaussian quadrature rules, we immediately deduce the two-dimensional quadrature rules, by successively integrating the function with respect to the first and the second variable. The numerical integration formulas obtained by this approach are called tensor-product quadrature. If $\left(t_{i,n+1}\right)_{0\leq i\leq n}, \left(t_{i,m+1}\right)_{0\leq i\leq m}, \left(a_{i,n+1}\right)_{0\leq i\leq n}, \left(a_{i,m+1}\right)_{0\leq i\leq m}$ are respectively the nodes and the weights of two one-dimensional quadrature methods, then the corresponding tensorproduct quadrature rule, noticed $Q^{\otimes}_{n+1,m+1}$, is given by:

$$Q^{\otimes}_{n+1,m+1}[f] = \sum_{i=0}^{n}\sum_{j=0}^{m} w_{i,j,n+1,m+1} f\big(x_{i,j,n+1,m+1}\big), \qquad (61)$$

where, for all $i = 0, \dots, n, j = 0, \dots, m$ :

$$\begin{aligned} \boldsymbol{x}_{i,j,n+1,m+1} &= \big(t_{i,n+1}, t_{j,m+1}\big) \\ w_{i,j,n+1,m+1} &= a_{i,n+1}a_{j,m+1}. \end{aligned} \qquad (62)$$

1) Accuracy of the tensor-product quadrature for $\mathbb{S}_0$ : For the sake of simplicity, we assume here $w(\boldsymbol{x}) = 1$. For fixed $n \in \mathbb{N}$, we are now interested in the approximation $I[f]$ by $Q^{\otimes}_{n+1,m+1}[f]$ given by Relation (61). Assume that $m \le n$, and that $f \in \mathcal{C}^{n+1}(\mathbb{S}_0)$. Let $M_{n+1}$ the upper bound of all the $(n+1)$-th derivatives of $f$ over $\mathbb{S}_0$ :

$$M_{n+1} = \max_{(x_1,x_2)\in\mathbb{S}_0}\left|\frac{\partial^{n+1} f(x_1,x_2)}{\partial x_1^{n+1-i}\partial x_2^{i}}\right|. \qquad (63)$$

Let $f_1$ and $f_2$ be the univariate functions defined as follows:

$$\begin{aligned} f_1(x_2) &= \int_{-1}^{1} f(x_1,x_2)dx_1, \\ f_2(x_1) &= \int_{-1}^{1} f(x_1,x_2)dx_2. \end{aligned} \qquad (64)$$

Since $f \in \mathcal{C}^{n+1}(\mathbb{S}_0)$, then $f_1 \in \mathcal{C}^{n+1}([-1,1])$ and $f_2 \in \mathcal{C}^{n+1}([-1,1)$. On the other hand, using the $n+1$-node quadrature rule to approximate $f_1(x_2)$ by $\tilde{f}_1(x_2)$, there exists a function $\xi_1$ of $x_2$, such that:

$$\tilde{f}_1(x_2) = \sum_{i=0}^{n} a_{i,n+1} f\big(t_{i,n+1}, x_2\big) = \int_{-1}^{1} f(x_1,x_2)dx_1 - \xi_1(x_2), \qquad (65)$$

Since $f \in \mathcal{C}^{n+1}(D)$, and $f_1 \in \mathcal{C}^{n+1}([-1,1])$, then:

$$f\big(t_{i,n+1}, x_2\big) \in \mathcal{C}^{n+1}([-1,1]), \forall i = 0,.\ \ldots, n \xi_1 \in \mathcal{C}^{n+1}([-1,1]) \qquad (66)$$

According to Relation (33), there exists a positive real constant $K_1$, such that:

$$|\xi_1(x_2)| \le \frac{K_1}{(n+1)!}, \forall x_2 \in [-1,1]. \qquad (67)$$

For all $i = 0, \dots, n$, using the $m+1$-node quadrature rule to approximate the integral of $f\big(t_{i,n+1}, x_2\big)$ over $[-1,1]$ by $\tilde{f}_2\big(t_{i,n+1}\big)$, there exists a function $\xi_2$ of $t_{i,n+1}$, such that:

$$\tilde{f}_2\big(t_{i,n+1}\big) = \sum_{j=0}^{m} a_{j,m+1} f\big(t_{i,n+1}, t_{j,m+1}\big) = \int_{-1}^{1} f\big(t_{i,n+1}, x_2\big)dx_2 + \xi_2\big(t_{i,n+1}\big). \qquad (68)$$

According to relation (33), there exists a positive real constant $K_2$ (independent of $i$ ), such that:

$$\big|\xi_2\big(t_{i,n+1}\big)\big| \le \frac{K_2}{(m+1)!}, \forall i = 0, \dots, n. \qquad (69)$$

By integrating the two members of relation (65) with respect to $x_2$, and by using (68), we obtain:

$$\begin{aligned} \sum_{i=0}^{n} a_{i,n+1}\int_{-1}^{1} f\big(t_{i,n+1}, x_2\big)dx_2 &= \int_{-1}^{1}\int_{-1}^{1} f(x_1,x_2)dx_1dx_2 - \int_{-1}^{1}\xi_1(x_2)dx_2, \\ &= I[f] - \int_{-1}^{1}\xi_1(x_2)dx_2. \end{aligned} \qquad (70)$$

On the other hand,

$$\begin{aligned} \sum_{i=0}^{n} a_{i,n+1}\int_{-1}^{1} f\big(t_{i,n+1}, x_2\big)dx_2 &= \sum_{i=0}^{n} a_{i,n+1}\left[\sum_{j=0}^{m} a_{j,m+1} f\big(t_{i,n+1}, t_{j,m+1}\big) + \int_{-1}^{1}\xi_2\big(t_{i,n+1}\big)dx_2\right] \\ &= Q^{\otimes}_{n+1,m+1}[f] + 2\sum_{i=1}^{n} a_{i,n+1}\xi_2\big(t_{i,n+1}\big), \end{aligned}$$

(71)

By combining the relations (70) and (71), the absolute error $E_{n+1,m+1}$, resulting from the approximation of $I[f]$ by $Q^{\otimes}_{n+1,m+1}[f]$ is given by:

$$\begin{aligned} E_{n+1,m+1} &= \left|I[f] - Q^{\otimes}_{n+1,m+1}[f]\right| \\ &= \left|\int_{-1}^{1} \xi_1(x_2)dx_2 + 2\sum_{i=0}^{n} a_{i,n+1}\xi_2(t_{i,n+1})\right|, \\ &\leq \sup_{[-1,1]} |\xi_1(x_2)| \int_{-1}^{1} dx_2 + 2\sup_{[-1,1]} |\xi_2(x_1)| \sum_{i=0}^{n} a_{i,n+1} \\ &\leq 2\frac{K_1}{(n+1)!} + 2\frac{K_2}{(m+1)!}S \text{ where } S = \sum_{i=0}^{n} a_{i,n+1}. \end{aligned} \tag{72}$$

To determine the value of $S$, it suffices to apply the one-dimensional $n$ +1-node quadrature rule on the constant function $\delta(t) = 1$ :

$$\int_{-1}^{1} \delta(t)dt = \sum_{i=0}^{n} a_{i,n+1} = 2 \tag{73}$$

Thus, the minimum value of the absolute error $E_{n+1,m+1}$ is reached $m = n$, and:
Theorem 4.2:

$$E_{n+1,n+1} = \left|I[f] - Q^{\otimes}_{n+1,n+1}\right| \leq \frac{K}{(n+1)!}, \text{ where } K = 2(K_1 + 2K_2). \tag{74}$$

Relation (74) means that the tensor-product quadrature rule requires $(n+1)^2$ nodes to have a convergence rate equal to that of the one-dimensional $n+1$-node quadrature rule. Thus, to optimize the tensor-product quadrature rule, it is wise to consider only the rules of the form $Q_{m+1,m+1}$. In the following, we will only consider this type of tensor-product rules. Moreover, to simplify notations, we will write:

1) $Q^{\otimes}_{n+1}$ to denote $Q^{\otimes}_{n+1,n+1}$,
2) $\boldsymbol{x}_{i,j,n+1}$ to denote $(x_{i,n+1}, x_{j,n+1})$,
3) $w_{i,j,n+1}$ to denote $w_{i,n+1}w_{j,n+1}$.

2) Strength of the tensor-product quadrature rule for $\mathbb{S}_0$ : Recall that for all $n \in \mathbb{N}$, $\mathbb{P}_{2,n} = \mathbb{R}_n[X,Y]$ is the vector space of polynomials in $x$ and $y$, whose degrees are less than or equal to $n$. We introduce in Definition 4.1, a novel set of polynomials, called tensor-product vector space, and noticed $\mathbb{P}^{\otimes}_{2,n}$.

Definition 4.1 (tensor-product polynomials): The set

$$\mathbb{P}^{\otimes}_{2,n} = \{P_1(x)P_2(y) \mid P_1 \text{ and } P_2 \in \mathbb{P}_{1,n}\}. \tag{75}$$

is a vector subspace of $\mathbb{P}_{2,2n}$ formed by all bivariate polynomials of degree at most $n$ in $x$ and $y$ separately.

Remark 4.2: Based on Definition 4.1, it is obvious that for all $n \in \mathbb{N}$,

$$\mathbb{P}_{2,n} \subset \mathbb{P}^{\otimes}_{2,n} \subset \mathbb{P}_{2,2n}. \tag{76}$$

Proposition 4.1 (strengnth of tensor-product quadrature rule): Let $n \in \mathbb{N}$, and $Q^{\otimes}_{n+1}[f]$ be the tensor-product quadrature approximation of $I[f]$. Then

$$Q^{\otimes}_{n+1}[f] = I[f] \Leftrightarrow f \in \mathbb{P}^{\otimes}_{2,2n+1}, \tag{77}$$

**Proof :** Let $f_1$, and $f_2$ the functions defined by (64). Since $f \in\in \mathbb{P}^{\otimes}_{2,2n+1}$, then, $f_1 \in \mathbb{P}_{1,2n+1}$, and $f_2 \in \mathbb{P}_{1,2n+1}$. Therefore, for all $i = 0, \ldots, n$, we have:

$$\begin{aligned} f_2(x_{i,n+1}) &= \int_{-1}^{1} f(x_{i,n+1}, x_2)dx_2 = \sum_{j=0}^{n} w_{j,n+1} f(x_{i,n+1}, x_{j,n+1}) \\ \int_{-1}^{1} f_2(x_1)dx_1 &= \sum_{i=0}^{n} w_{i,n+1} f_2(x_{i,n+1}) \end{aligned} \tag{78}$$

On the other hand, using (78), we have:

$$
\begin{aligned}
Q_{n+1}[f] &= \sum_{i=0}^{n}\sum_{j=0}^{n} w_{i,n+1}w_{j,n+1}f\left(x_{i,n+1},x_{j,n+1}\right)\\
&= \sum_{i=0}^{n} w_{i,n+1}\left[\sum_{j=0}^{n} w_{j,n+1}f\left(x_{i,n+1},x_{j,n+1}\right)\right]\\
&= \sum_{i=0}^{n} w_{i,n+1}f_2\left(x_{i,n+1}\right)\\
&= \int_{-1}^{1} f_2(x_1)dx_1\\
&= \int_{-1}^{1}\int_{-1}^{1} f(x_1,x_2)dx_2dx_1 = I[f]
\end{aligned}
\tag{79}
$$

Assume now that $f(x,y)=q(x)=x^{2n+2}$. It is clear that $f\notin \mathbb{P}_{2,2n+1}^{\otimes}$. Moreover, according to 1-D quadrature rule properties stated in Section III, we have:

$$
\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy &= 2\int_{-1}^{1} x^{2n+2}dx\\
&= 2\sum_{i=0}^{n+1} w_{i,n+2}q\left(x_{i,n+2}\right),\\
&\neq 2\sum_{i=0}^{n} w_{i,n+1}q\left(x_{i,n+1}\right).
\end{aligned}
\tag{80}
$$

We deduce that:

$$
\begin{cases}
I[f]=Q_{n+1}^{\otimes}[f] & \text{if,}\quad f\in \mathbb{P}_{2,2n+1}^{\otimes}\\
I[f]\neq Q_{n+1}^{\otimes}[f] & \text{if,}\quad f\notin \mathbb{P}_{2,2n+1}^{\otimes}.
\end{cases}
\tag{81}
$$

## V. QUADRATURE RULES FOR THE TRIANGLE

In a barycentric coordinate system, any symmetry that leaves the triangle invariant is defined as a permutation of the barycentric coordinates. These symmetries are easy to identify on an equilateral triangle $\mathbb{T}=[A;B;C]$. Indeed, in the latter case, they are defined as follows:

1) the identity map $Id_2$,
2) the symmetries with respect to the medians of segments $[A,B]$, $[B,C]$, and $[C,A]$,
3) the rotations with center $G$ and angles $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$, where $G$ is the centroid of $\mathbb{T}$. A point $M$ in the triangle $\mathbb{T}_0$, can have one of the following barycentric coordinates

$$
\begin{aligned}
&\text{type-1:} && M=\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right),\\
&\text{type-2:} && M=(\alpha_1,\alpha_1,1-2\alpha_1),\\
&\text{type-3:} && M=(\alpha_1,\alpha_2,1-\alpha_1-\alpha_2).
\end{aligned}
\tag{82}
$$

Let $M$ be a point inside the triangle whose barycentric coordinates are $(\alpha_1,\alpha_2,\alpha_3)$. Let $\mathcal{O}(M)$ be the orbit generated by $M$, then $\mathcal{O}(M)$ consists of all the points whose barycentric coordinates are permutations of the triplet $(\alpha_1,\alpha_2,\alpha_3)$. Therefore,

1) if $M$ is of type-1, then $card\ (\mathcal{O}(M))=1$,
2) if $M$ is of type-2, then $card\ (\mathcal{O}(M))=3$,
3) if $M$ is of type-3, then $card\ (\mathcal{O}(M))=6$,

Thus, a $n+1$-node quadrature rule consists of $n_1$ type-1 orbits, $n_2$ type-2 orbits and $n_3$ type-3 orbits such that,

$$n+1=n_1+3n_2+6n_3. \tag{83}$$

Since there is a unique type- 1 orbit, then $n_1\in\{0,1\}$, and

$$n+1=3n_2+6n_3 \ \text{ or } \ n=3n_2+6n_2. \tag{84}$$

Knowing that,

1 for a given orbit, the weights of its nodes are identical [21],

2 the elements of a given type- 2 orbit require a single parameter $\alpha_1$,

3 the elements of a given type-3 orbit require a single parameter $\alpha_1$,

then a symmetric $n+1$-node quadrature rule requires $n_1+2n_2+3n_3$ parameters,

1) $n_1+n_2+n_3$ parameters to define the weights,
2) $n_2+2n_3$ parameters to define the coordinates of the nodes.

Thus, to determine a $n+1$-node symmetric quadrature rule, we must first decompose $n+1$ into a triplet $(n_1,n_2,n_3)$, then rewrite system (17) or criterion (28), taking into account the conditions mentioned in Corollary 2.1.

## A. Quadrature rule for arbitrary triangles

Let $O=(0{,}0), B=(1{,}0)$ and $C=(0{,}1)$ be the vertices of the triangle $\mathbb{T}_0=[O;B;C]$. Let $\mathbb{T}'$ be an arbitrary triangle $[A';B';C']$,such that:

$$A'=(a_1,a_2)^t\ ;\ B'=(b_1,b_2)^t\ ;\ C'=(c_1,c_2)^t. \qquad (85)$$

Let $F$ be the affine map, which transforms $\mathbb{T}_0$ into $\mathbb{T}'$. Then,

$$F(O)=A'\ ;\ F(B)=B'\ ;\ F(C)=C'\ ;\ F(\mathbb{T}_0)=\mathbb{T}' \qquad (86)$$

Let $M=(x,y)^t\in\mathbb{T}_0$, and $M'=(x',y')^t=F(M)$. Then,

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}b_1-a_1 & c_1-a_1\\ b_2-a_2 & c_2-a_2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}a_1\\a_2\end{pmatrix}=P\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}a_1\\a_2\end{pmatrix}$$

where,

$$P=\begin{pmatrix}b_1-a_1 & c_1-a_1\\ b_2-a_2 & c_2-a_2\end{pmatrix} \qquad (87)$$

Since,

$$|det\ (P)|=2\,Area(\mathbb{T}'), \qquad (88)$$

then, by using the change of variables formula, we get:

$$\iint_{\mathbb{T}'} f(x',y')dx'dy' = \iint_{\mathbb{T}_0} f(F(x,y)^t)|det\ (P)|dxdy \qquad (89)$$

By approximating $I[f\circ F]$ by $Q_{n+1}[f\circ F]$, we get,

$$\iint_{\mathbb{T}'} f(x',y')dx'dy' \simeq 2Area\ (\mathbb{T}')\sum_{i=0}^{n} w_{i,n+1}f\left(F(x_{i,n+1},y_{i,n+1})\right) = \sum_{i=0}^{n} w'_{i,n+1}f(x'_{i,n+1},y'_{i,n+1}) \qquad (90)$$

where, for all $i=0,\dots,n$,

$$w'_{i,n+1}=2Area\ (\mathbb{T}')w_{i,n+1}\ \text{ and }\ (x'_{i,n+1},y'_{i,n+1})^t=F\left((x_{i,n+1},y_{i,n+1})^t\right). \qquad (91)$$

In the following, to avoid overloading the notations, we write:

$$Q_{n+1}[f]=\sum_{i=0}^{n} w_{i,n+1}f(\boldsymbol{x}_{i,n+1}) \qquad (92)$$

to represent the $n+1$-node quadrature rule on any triangle $\mathbb{T}'$ keeping in mind relation (91).

## B. Accuracy of the quadrature rule for the triangle

Theorem 5.1: Let $\mathbb{T}$ be a triangle whose largest edge length equals $h$. Let $E_{n+1}[f]$ be the error made in approximating $I[f]$ by $Q_{n+1}[f]$.

$$E_{n+1}[f]=\iint_{\mathbb{T}} f(x,y)dxdy - \sum_{i=0}^{n} w_{i,n}f(x_{i,n+1},y_{i,n+1}). \qquad (93)$$

where,

If $f\in\mathcal{C}^{m+1}(\mathbb{T})$, then there a exists a positive constant $M_{m+1}$ such that:

$$|E_{n+1}[f]|\le M_{m+1}\frac{2^{m+1}}{(m+1)!}h^{m+3}, \qquad (94)$$

where $m$ is the strength of $Q_{n+1}$.

Proof : Let $(x_0,y_0)$ be a point inside the open triangle $\mathbb{T}$. The Taylor-Lagrange expansion of $f$ at $(x_0,y_0)$, ensure the existence of a point $(\lambda,\theta)\in\mathbb{T}$, a polynomial $P_m\in\mathbb{P}_{2,m}$, and a polynomial $R_m\in\mathbb{P}_{2,m+1}$, such that:

$$f(x,y)=P_m(x,y)+R_m(x,y), \qquad (95)$$

$$R_m(x,y) = \frac{1}{(m+1)!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^{m+1} f(\lambda,\theta)$$
$$= \frac{1}{(m+1)!}\left[\sum_{i=1}^{m+1}\binom{m+1}{i}(x-x_0)^{m+1-i}(y-y_0)^i \frac{\partial^{m+1} f(\lambda,\theta)}{\partial x^{m+1-i}\partial y^i}\right]$$

Since $P_m \in \mathbb{P}_{2,m}$, then,

$$E_{n+1}[f] = \iint_{\mathbb{T}} P_m(x,y)dxdy + \iint_{\mathbb{T}} R_k(x,y)dxdy - Q_{n+1}[P_m] - Q_{n+1}[R_m],$$
$$= \frac{1}{(m+1)!}\iint_{\mathbb{T}}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^{m+1} f(\lambda,\theta)dxdy - Q_{n+1}[R_m], \quad (96)$$

where,

$$Q_{n+1}[R_m] = \sum_{i=1}^{n} w_{i,n+1}\iint_{\mathbb{T}}\left[\left(x_{i,n+1}-x_0\right)\frac{\partial}{\partial x}+\left(y_{i,n+1}-y_0\right)\frac{\partial}{\partial y}\right]^{k+1} f(\lambda_i,\theta_i)dxdy. \quad (97)$$

Since $(\lambda,\theta)$ depends on $(x,y)$, then $(\lambda_i,\theta_i)$ depend also on $\left(x_{i,n+1},y_{i,n+1}\right)$. Let $M_{m+1}$ be the upper bound of all the $(m+1)$ derivatives of $f$ over $T$ :

$$M_{m+1} = \max_{(x,y)\in\mathbb{T}}\left|\frac{\partial^{m+1} f(x,y)}{\partial x^{m+1-i}\partial y^i}\right|. \quad (98)$$

On the other hand, we have:

$$\left|\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^{m+1} f(x,y)\right| \le M_{m+1}\left[\sum_{i=1}^{m+1}\binom{m+1}{i}|x-x_0|^{m+1-i}|y-y_0|^i\right]$$
$$\le M_{m+1}(|x-x_0|+|y-y_0|)^{m+1}$$
$$\le M_{m+1}(2h)^{m+1}. \quad (99)$$

Since $1 \in \mathbb{P}_{2,m+1}$, then,

$$\iint_{\mathbb{T}} dxdy = Area(\mathbb{T}) = \sum_{i=0}^{n} w_{i,n+1}. \quad (100)$$

Since $Area\,(\mathbb{T}) \le h^2/2$, then we deduce from (98), (99), and (100) that,

$$|E_{n+1}| \le \frac{M_{m+1}}{(m+1)!}\left[\iint_{\mathbb{T}}(2h)^{m+1}dxdy + \sum_{i=0}^{n} w_{i,n}(2h)^{m+1}\right]$$
$$\le \frac{M_{m+1}}{(m+1)!}(2h)^{m+1}\left[Area\,(\mathbb{T}) + \sum_{i=0}^{n} w_{i,n}\right] \quad (101)$$
$$\le \frac{M_{m+1}}{(m+1)!}(2h)^{m+1}[2Area\,(\mathbb{T})]$$
$$= M_{m+1}\frac{2^{m+1}}{(m+1)!}h^{m+3}.$$

Remark 5.1: Note that the upper bound given by relation (94) was calculated under the assumption $f \in \mathcal{C}^{m+1}(\mathbb{T})$. In practice, if $f$ were of class $\mathcal{C}^{\kappa+1}(\mathbb{S})$, with $\kappa < m$, then the error $E_{n+1}$ would be bounded from above by $M_{\kappa+1}2^{\kappa+1}h^{\kappa+3}/(\kappa+1)$ .

## VI. QUADRATURE FOR THE TRIANGLE AND THE SQUARE BY CHANGE OF VARIABLES

A. Quadrature for the triangle

Let $\mathring{\mathbb{T}}_0$ (resp.. $\mathring{\mathbb{S}}_0$ ) be the open interior of $\mathbb{T}_0$ $(resp.\,\mathbb{S}_0$

$$\overset{\circ}{\mathbb{T}}_0 = \{(x,y) \in \mathbb{R}^2 \mid x,y \in ]0,1[ \text{ and } x+y \in ]0,1[\},$$
$$\overset{\circ}{\mathbb{S}}_0 = \{(x,y) \in \mathbb{R}^2 \mid x,y \in ]-1,1[[\,^2\}. \quad (102)$$

Theorem 6.1: The mapping,

$$\Phi: \overset{\circ}{\mathbb{T}}_0 \longrightarrow \overset{\circ}{\mathbb{S}}_0$$

$$(x,y) \mapsto \begin{cases} s & \longrightarrow 1 + x - y - \sqrt{(x-y)^2 + 4(1-x-y)} \\ t & = 1 - x + y - \sqrt{(x-y)^2 + 4(1-x-y)} \end{cases} \quad (103)$$

is a diffeomorphism which transforms $\overset{\circ}{\mathbb{T}}_0$ into $\overset{\circ}{\mathbb{S}}_0$. Moreover,

$$\Phi^{-1} : \overset{\circ}{\mathbb{S}}_0 \longrightarrow \overset{\circ}{\mathbb{T}}_0$$

$$(s,t) \mapsto \begin{cases} x = \dfrac{(1+s)(3-t)}{8}, \\ y = \dfrac{(3-s)(1+t)}{8}. \end{cases} \quad (104)$$

Lemma 6.1: Let $f$ be a continuous function on the reference triangle $\mathbb{T}_0$. Then,

$$\iint_{\mathbb{T}_0} f(x,y)dxdy = \frac{1}{16}\iint_{\mathbb{S}_0} f\left(\frac{(1+s)(3-t)}{8}, \frac{(3-s)(1+t)}{8}\right)(2-s-t)dsdt \quad (105)$$

Proof : The Jacobian determinant of $\Phi^{-1}$ at the point $(s,t)$ is given by:

$$J(\Phi^-1)(s,t) = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix} = \frac{|2-s-t|}{16} = \frac{(2-s-t)}{16} \quad (106)$$

Using the change of variables formula, we get:

$$\begin{aligned} \iint_{\mathbb{T}_0} f(x,y)dxdy &= \iint_{\mathbb{T}_0} f(x,y)dxdy \\ &= \iint_{\mathbb{S}_0} f(\Phi^{-1}(s,t))|J(\Phi^{-1})(s,t)|dsdt \\ &= \iint_{\mathbb{S}_0} f(\Phi^{-1}(s,t))|J(\Phi^{-1})(s,t)|dsdt \\ &= \iint_{\mathbb{S}_0} f\left(\frac{(1+s)(3-t)}{8}, \frac{(3-s)(1+t)}{8}\right)\frac{(2-s-t)}{16}dsdt \end{aligned}$$

Proposition 6.1: For all $n \in \mathbb{N}$, we denote by $Q_{n+1}^S$ the quadrature rule on the rectangle $\mathbb{S}_0$ whose nodes $(s_{i,n+1}, t_{i,n})_{0 \le i \le n}$ and the weights are $(w_{i,n+1})_{0 \le i \le n}$. Let $Q_{n+1}^T$ be the quadrature rule on the triangle $\mathbb{T}_0$ whose nodes $(x_{i,n+1}, y_{i,n+1})_{0 \le i \le n}$, and weights $(a_{i,n+1})_{0 \le i \le n}$ are defined by:

$$\begin{aligned} x_{i,n+1} &= \frac{(1+s_{i,n+1})(3-t_{i,n+1})}{8} \\ y_{i,n+1} &= \frac{(3-s_{i,n+1})(1+t_{i,n+1})}{8} \\ a_{i,n+1} &= \frac{2-s_{i,n+1}-t_{i,n+1}}{16} w_{i,n+1} \end{aligned} \quad (107)$$

Denote by,

$$r_m = \left\lceil \frac{m-1}{2} \right\rceil, \quad (108)$$

where $m$ is the strength of $Q_{n+1}^S$, and $\lceil\cdot\rceil$ is the ceil symbol. Then,

$$\begin{aligned} \iint_{\mathbb{T}_0} f(x,y)dxdy &\approx Q_{n+1}^T[f] = \sum_{i=0}^{n} a_{i,n+1} f(x_{i,n+1}, y_{i,n+1}), \\ \iint_{\mathbb{T}_0} f(x,y)dxdy &= Q_{n+1}^T[f] \quad \text{if } f \in \mathbb{P}_{2,r_m}, \end{aligned} \quad (109)$$

**Proof :** Let $g$ be an integrable function over $\mathbb{S}_0$. Then, for all $n \in \mathbb{N}$, we have:

$$\iint_{\mathbb{S}_0} g(s,t)dsdt \approx Q_{n+1}^S[g] = \sum_{i=0}^{n} w_{i,n+1} g(s_{i,n+1}, t_{i,n+1}). \quad (110)$$

By combining relations (105) (110), we easily get:

$$\iint_{\mathbb{T}_0} f(x,y)dxdy \approx \sum_{i=0}^{n} a_{i,n+1} f(x_{i,n+1}, y_{i,n+1}) = Q_{n+1}^{T}[f].$$

Remark that for all $m \in \mathbb{N}, 2r_m = m-1$. Therefore, if $f \in \mathbb{P}_{2,r_m}$, then,

$$f\left(\frac{(1+s)(3-t)}{8}, \frac{(3-s)(1+t)}{8}\right) \quad \in \quad \mathbb{P}_{2,2r_m} = \mathbb{P}_{2,m-1},$$

and $(2-s-t)f\left(\frac{(1+s)(3-t)}{8}, \frac{(3-s)(1+t)}{8}\right) \quad \in \quad \mathbb{P}_{2,m}$.

By construction, $I[f] = Q_{n+1}^{S}[f]$ if the integrand belongs to $\mathbb{P}_{2,m}$. Therefore, $Q_{n+1}^{T}$ is of strength $r_m$. Moreover, since $Q_{n+1}^{S}[f]$ is PIS and the transformation (107) preserves the properties of $Q_{n+1}^{S}$, then $Q_{n+1}^{T}$ is also PIS.

B. Quadrature for the square

1) Quadrature rule by diffeomorphism $\Phi$ : For all $n \in \mathbb{N}$, we denote by $Q_{n+1}^{T}$ the quadrature rule on the triangle $\mathbb{T}_0$ whose nodes $(x_{i,n+1}, y_{i,n})_{0\le i\le n}$ and the weights are $(a_{i,n+1})_{0\le i\le n}$. Let $Q_{n+1}^{S}$ be the quadrature rule on the rectangle $\mathbb{S}_0$ whose nodes $(s_{i,n+1}, t_{i,n+1})_{0\le i\le n}$, and weights $(w_{i,n+1})_{0\le i\le n}$ are defined by:

$$\begin{aligned} s_{i,n+1} &= 1 + x_{i,n+1} - y_{i,n+1} - \sqrt{(x_{i,n+1} - y_{i,n+1})^2 + 4(1 - x_{i,n+1} - y_{i,n+1})}, \\ t_{i,n+1} &= 1 - x_{i,n+1} + y_{i,n+1} - \sqrt{(x_{i,n+1} - y_{i,n+1})^2 + 4(1 - x_{i,n+1} - y_{i,n+1})}, \\ w_{i,n+1} &= \frac{16}{2 - x_{i,n+1} - y_{i,n+1}} a_{i,n+1}. \end{aligned} \qquad (111)$$

Then, the formula:

$$\iint_{\mathbb{S}_0} f(x,y)dxdy \approx Q_{n+1}^{S}[f] = \sum_{i=0}^{n} w_{i,n+1} f(s_{i,n+1}, t_{i,n+1}) \qquad (112)$$

constitutes a PIS quadrature rule on $\mathbb{S}_0$.

Remark 6.1: The quadrature rule (112) has two main drawbacks. The first is caused by the rational fraction $\frac{1}{2-x-y}$, which tends to infinity at $(x,y) = (1,1)$. The second is related to the indeterminacy of the strength of this rule.

2) Quadrature rule by diagonal division of the square: Let $A_1 = (-1,-1), A_2 = (1,-1), A_3 = (1,1)$, and $A_4 = (-1,1)$ be the vertices of the square $\mathbb{S}_0 = [-1,1]^2$. Denote by $\mathbb{T}_1$ the triangle $[A_1, A_2, A_4]$, and $\mathbb{T}_2$ the triangle $[A_3, A_4, A_2]$.

1) By using the quadrature rules on the arbitrary triangles given by relation (87), we obtain the nodes $(s_{i,n+1}^1, t_{i,n+1}^1)_{0\le i\le n}$, and the weights $(w_{i,n+1}^1)_{0\le i\le n}$, associated with $\mathbb{T}_1$ :

$$\begin{pmatrix} s_{i,n+1}^1 \\ t_{i,n+1}^1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_{i,n+1} \\ y_{i,n+1} \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ and } w_{i,n+1}^1 = 4w_{i,n+1.} \qquad (113)$$

2) By using the quadrature rules on the arbitrary triangles given by relation (87), we obtain the nodes $(s_{i,n+1}^2, t_{i,n+1}^2)_{0\le i\le n}$, and the weights $(w_{i,n+1}^2)_{0\le i\le n}$, associated with $\mathbb{T}_2$:

$$\begin{pmatrix} s_{i,n+1}^2 \\ t_{i,n+1}^2 \end{pmatrix} = -\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_{i,n+1} \\ y_{i,n+1} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ and } w_{i,n+1}^2 = 4w_{i,n+1}, \qquad (114)$$

where $(x_{i,n+1}, y_{i,n+1})_{0\le i\le n}$, and $(w_{i,n+1})_{0\le i\le n}$, are respectively the nodes and weights associated with the reference triangle $\mathbb{T}_0$.

Proposition 6.2: For all $n \in \mathbb{N}$,

$$\begin{aligned} \iint_{\mathbb{S}_0} f(x,y)dxdy &\approx Q_{2n+2}^{T}[f] \\ &= \sum_{i=0}^{n} w_{i,n+1}^1 f(s_{i,n+1}^1, t_{i,n+1}^1) + \sum_{i=0}^{n} w_{i,n+1}^2 f(s_{i,n+1}^2, t_{i,n+1}^2), \quad , \qquad (115) \\ \iint_{\mathbb{S}_0} f(x,y)dxdy &= Q_{2n+2}, \text{ if } f \in \mathbb{P}_{2,m}, \end{aligned}$$

where $m$ is the strength of $Q_{n+1}^{T}$.

**Proof :** Since $\mathbb{S}_0 = \mathbb{T}_1 \cup \mathbb{T}_2$, and $\mathbb{T}_1 \cap \mathbb{T}_2 = \emptyset$, then,

$$\begin{aligned}\iint_{\mathbb{S}_0} f(x,y)dxdy &= \iint_{\mathbb{T}_1} f(x,y)dxdy + \iint_{\mathbb{T}_2} f(x,y)dxdy \\ &\approx \sum_{i=0}^{n} w_{i,n+1}^1 f\left(s_{i,n+1}^1, t_{i,n+1}^1\right) + \sum_{i=0}^{n} w_{i,n+1}^2 f\left(s_{i,n+1}^2, t_{i,n+1}^2\right)\end{aligned} \quad (116)$$

By construction, $I[f] = Q_{n+1}^T\,[f]$ if the integrand belongs to $\mathbb{P}_{2,m}$. Thus, if $f \in \mathbb{P}_{2,m}$, we get:

$$\begin{aligned}\iint_{\mathbb{T}_1} f(x,y)dxdy &= \sum_{i=0}^{n} w_{i,n+1}^1 f\left(s_{i,n+1}^1, t_{i,n+1}^1\right) \\ \iint_{\mathbb{T}_2} f(x,y)dxdy &= \sum_{i=0}^{n} w_{i,n+1}^2 f\left(s_{i,n+1}^2, t_{i,n+1}^2\right)\end{aligned} \quad (117)$$

Therefore, $Q_{2n+2}^S$ is of strength $m$. Moreover, since $Q_{n+1}^T\,[f]$ is PIS then $Q_{2+2}^S$ is also PIS.

## VII. EFFICIENT PIS QUADRATURE RULES FOR THE SQUARE AND THE TRIANGLE

After summarizing the different approaches that have made it possible to construct PIS quadrature rules for the square and the triangle, we give a historical overview of those which are efficient. These rules classified first according to their strength, then according to their date of publication.

A. Efficient PIS quadrature rules for the square

- In 1908, Burnside [27] used one type-2 orbit and one type-3 orbit to determine a quadrature rule of strength 5. This rule uses only 8 nodes and is therefore considered optimal.

- In 1953, Tyler [28] succeeded in obtaining a quadrature rule of strength 7, using one type-2 orbit and two type-3 orbits. This rule only requires 12 nodes and is therefore considered optimal.

- Rabinowitz proves in his article [29] that it is not possible to obtain a quadrature rule of strength greater than or equal to 8 without including type- 4 orbits. He also proves in the same article, that for a given strength $m \geq 8$, an optimal quadrature rule must minimize the number of type-4 orbits. The approach followed by Rabinowitz allowed him to find quadrature rules of strengths up to 15. The quadrature rules obtained by Robinowitz are those which use the fewest nodes among all the PIS quadrature rules.

  Unlike the PIS quadrature rules of strength less than or equal to 15, all PIS quadrature rules of strength greater than or equal to 16 have been obtained as an approximate solution to a system of nonlinear equations (17), or to a nonlinear least-squares problem (28).

- In 1985 Duvavant [30] transforms the system of nonlinear equations (17) into a problem of nonlinear least squares, to obtain a PIS quadrature rule common to strengths 16 and 17. The minimization of the least-squares criterion was carried out using the Levenberg-Marquardt finite-difference algorithm ([22], [31]).

- In 2015 Witherden et al [20] transform the system of nonlinear equations (19) into a problem of nonlinear least squares, to obtain a PIS quadrature rules for strengths 18 and 20. These rules are also valid for the strengths 19 and 21 respectively. The minimization of the least-squares criterion was also carried out using the Levenberg-Marquardt finite difference algorithm.

Table I presents the number of nodes corresponding to the optimal and efficient PIS quadrature rules for the square as a function of the strength m.

B. Efficient PIS quadrature rules for the triangle

Research on the quadrature rule for the triangle is more abundant than that for other geometric shapes. This is due to the fundamental role played by the triangle in the discretization of surfaces.

**TABLE I:** $n+1$ (resp. $n^*+1$ ) is the number of nodes for an efficient (resp. optimal) PIS quadrature rule of strength $m$.

| $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{n^*+1}$ | 3 | 4 | 6 | 7 | 10 | 12 | 15 | 17 | 21 | 24 |
| $\boldsymbol{n+1}$ | 4 | 4* | 8 | 8 | 12 | 12 | 20 | 20 | 28 | 28 |
| $\boldsymbol{m}$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| $\boldsymbol{n^*+1}$ | 28 | 31 | 36 | 40 | 45 | 49 | 55 | 60 | 66 | 71 |
| $\boldsymbol{n+1}$ | 37 | 37 | 48 | 48 | 60 | 60 | 72 | 72 | 85 | 85 |

using triangulation techniques. The search for quadrature rules on the triangle has been studied for more than a century, and is still relevant today. This study is restricted to efficient quadrature rules satisfying all desirable properties, namely efficient PIS rules. Here we present a historical overview of these rules and order them according to their strengths, followed by their publication dates.

- Using orthogonal Jacobi polynomials $\left(P_n^{[\alpha,\beta]}\right)_{n\geq 0}$ on the interval $[0,1]$ and orthogonal Legendre polynomials on the interval $[-1,1]$, Hammer et al [32] succeeded, in 1956, to exhibit a PIS quadrature rule of strength $m$, using only $2m-1$ nodes. Their rule is only optimal for $m \leq 2$.
- Writing the system of equations (17) in barycentric coordinates, Albrecht et al [33] proposed in 1958 a 6-node PIS quadrature rule of strength 3. This rule is efficient but not optimal, since the lower bound is 4.
- In 1973, Cowper [34] used two type-2 orbits to determine a PIS quadrature rule of strength 4. This rule uses only 6 nodes and is therefore optimal.
- In 1948, Radon [35] used one type-1 orbit and two type-2 orbits, and barycentric coordinates to exhibit a PIS quadrature rule of strength 5. This rule uses only 7 nodes and is therefore optimal.
- In 1973, Cowper [34] used two type-2 orbits to determine a 12-node PIS quadrature rule of strength 6. This rule is efficient but not optimal, since the lower bound is 10.

Unlike PIS quadrature rules of strength less than or equal to 6, all PIS quadrature rules of strength greater than or equal to 7 have been obtained as an approximate solution to a system of nonlinear equations, or to a nonlinear least-squares problem.

- In 1978, Laursen et al prove in their article [36] that it is not possible to obtain a quadrature rule of strength greater than or equal to 7 without including type-3 orbits. They provide in their article [36], a 15-node PIS quadrature rule of strength 7, using one type-2 orbit and two type-2 orbits. This rule is efficient but not optimal, since the lower bound is 12.
- By writing the system of equations (17) in polar coordinates and exploiting the symmetry properties of the equilateral triangle $[(-1,0);(1/2,\sqrt{3}/2);(1/2,-\sqrt{3}/2)]$, Lyness et al [23] succeeded in reducing the complexity of the resulting nonlinear system. They showed in [23] that the determination of nodes, as well as their corresponding weights, was reduced to the search for zeros of univariate functions or polynomials. Their approach allowed them to find quadrature rules up to strength 11, which are symmetric but not necessarily PI. The rule corresponding to strength 8 (resp. 9) uses 16 (resp. 19) nodes and is efficient but not optimal, since the lower bound is 15 (resp. 17).
- By transforming the system of nonlinear equations (17) for the right triangle $[(0,0);(1,0);(0,1)]$, into a least squares problem, Zhang et al [37] succeeded in determining symmetric and positive quadrature rules up to strength 21 . The numerical resolution of this problem is performed using the subroutines provided by the MINPACK optimization package [38]. The rule corresponding to strength 10 (resp. 11 and 16) uses 25 (resp. 28 and 55) nodes and is efficient but not optimal, since the lower bound is 21 (resp. 24 and 45).
- In 1985, Dunavant [39] transformed the polar quadrature system generated by Lyness et al [36], into a least squares problem. The numerical resolution of this problem is carried out using the finite difference Levenberg-Marquardt routine (ZXSSQ), which contrary to routines ZSCNT and ZSPOW, is able to treat the nonlinear rectangular systems. His approach allowed him to find quadrature rules up to strength 20, which are symmetric but not necessarily PI. The rule corresponding to strength 12 (resp. 14 and 19) uses 33 (resp. 42 and 73) nodes and is efficient but not optimal, since the lower bound is 28 (resp. 36 and 60).

- In 1990, Berntsen et al [40] constructed a PIS quadrature rule of strength 13. Thus, they were the first to find an efficient rule for this strength using 36 nodes.
- In 2015 Witherden et al [20] adapted the system (19) to the triangle $[(-1,-1);(1,-1);(-1,1)]$, and transformed the resulting nonlinear system into a nonlinear least-squares problem, to obtain a PIS quadrature rules up to strength 20. The minimization of the objective functions is carried out using a Levenberg-Marquardt finite-difference algorithm. The rule corresponding to strength 15 uses 49 nodes and is efficient but not optimal, since the lower bound is 40.
- By adapting the system of nonlinear equations (19) to orthogonal polynomials on the equilateral triangle $[(-1,-1/\sqrt{3});(0,2/\sqrt{3});(1,-1/\sqrt{3})]$, Xiao et al [21] transform the resulting system into a least-squares problem. The minimization of the objective functions is carried out using least squares Newton's method. Their approach allowed them to find PIS quadrature rules up to strength 30. The rule corresponding to strength 17 (resp. 18, 20, and 21) uses 60 (resp. 67, 79, and 87) nodes and is efficient but not optimal, since the lower bound is 49 (resp. 55,66, and 71)

Table II presents the number of nodes corresponding to the optimal and efficient PIS quadrature rules for the triangle as a function of the strength $m$.

**TABLE II:** $n+1$ (resp. $n^*+1$ ) is the number of nodes for an efficient (resp. optimal) PIS quadrature rule of strength $m$.

| $\boldsymbol{m}$ | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** |
|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{n^*+1}$ | 3 | 4 | 6 | 7 | 10 | 12 | 15 | 17 | 21 | 24 |
| $\boldsymbol{n+1}$ | $3^*$ | 6 | $6^*$ | $7^*$ | 12 | 15 | 16 | 19 | 25 | 28 |
| $\boldsymbol{m}$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| $\boldsymbol{n^*+1}$ | 28 | 31 | 36 | 40 | 45 | 49 | 55 | 60 | 66 | 71 |
| $\boldsymbol{n+1}$ | 33 | 37 | 42 | 49 | 55 | 60 | 67 | 73 | 79 | 87 |

## VIII. NUMERICAL STUDY

The numerical study is carried out with MATLAB 2017. The nodes and their corresponding

weights are collected from their sources mentioned in sections VII-A and VII-B.

We test here the different numerical integration methods on the test functions introduced by Genz in [41]. These functions are divided into six classes. Each class poses a different challenge to numerical integration methods. The functions of the same class are characterized by one or two parameters. The first parameter denoted $a$, is composed of two positive reals $a_1$ and $a_2$ which control the integration difficulty. The second parameter denoted b, is composed of two reals $b_1, b_2 \in [0,1]$, which describe one of the attributes of the function such as the position of an extremum, of a discontinuity, etc. Table III presents the different classes of Genz functions, as well as their general expression. Table IV gives $I[f_i]$ for all $i = 1, . . . ,6$.

The Genz functions $f_1 \, . . . , f_6$, presented in Table III are defined on the square $\mathbb{S} = [0,1]^2$. For all, $i = 1, \ldots, 6$, we denote by $g_i$ the function defined on the square $\mathbb{S}_0$, by:

$$g_i(x,y) = \frac{1}{4} f_i\left(\frac{x+1}{2}, \frac{y+1}{2}\right). \qquad (118)$$

Remark 8.1:

1) Notice that for all $i = 1, \ldots, 6$, the function $g_i$ has the same properties as those of $f_i$. Moreover, we have:

$$\iint_{\mathbb{S}_0} g_i(x,y)dxdy = \iint_{\mathbb{S}} f_i(x,y)dxdy \qquad (119)$$

2) Figures 1, . . . , and 6 present the surfaces of $(g_i)_{1\le i\le 6}$, for the choice of parameters $\boldsymbol{a} = (2,10)$ and $\boldsymbol{b} = (0.8,0.2)$
3) This study is carried out only on the integrands $g_1$ , . . . , $g_6$, defined on the square $\mathbb{S}_0$.

**TABLE III:** Bivariate Genz test functions.

| Class | Expression of the functions |
|---|---|
| **Oscillatory** | $f_1(x,y) \quad = cos\,(2\pi b_1 + a_1 x + a_2 y)$ |
| **Continuous** | $f_2(x,y) \quad = exp\,(-a_1\lvert x-b_1\rvert + a_2\lvert y-b_2\rvert)$ |
| **Discontinuous** | $f_3(x,y) \quad = \begin{cases} 0 & \text{if } x > b_1 \text{ or } y > b_2 \\ exp\,(a_1 x + a_2 y) & \text{otherwise} \end{cases}$ |
| **Product Peak** | $f_4(x,y) \quad = \dfrac{a_1^2 a_2^2}{(1+a_1^2(x-b_1)^2)(1+a_2^2(y-b_2)^2)}$ |
| **Corner Peak** | $f_5(x,y) \quad = \dfrac{1}{(1+a_1 x + a_2 y)^3}$ |
| **Gaussian** | $f_6(x,y) \quad = exp\,(-a_2^2(x-b_1)^2 - a_2^2(y-b_2)^2)$ |

A. Graphic representation of integrands

B. Notations

In the following, we use the following abbreviation to designate the different numerical

methods:

**TABLE IV:** Exact integrals of Genz test functions.

| Class | Expression of the functions |
|---|---|
| **Oscillatory** | $I[f_1] = \dfrac{-cos\,(2\pi b_1) - cos\,(a_2 + a_1 + 2\pi b_1) + cos\,(2\pi b_1 + a_2) + cos\,(a_1 + 2\pi b_1)}{a_1 a_2}$ |
| **Continuous** | $I[f_2] = \prod_{i=1}^{2} \dfrac{2 - exp\,(-a_i b_i) - exp\,\left(a_i(b_i - 1)\right)}{a_i}$ |
| **Discontinuous** | $I[f_3] = \dfrac{1 - exp\,(a_1 b_1) - exp\,(a_2 b_2) + exp\,(a_1 b_1 + a_2 b_2)}{a_1 a_2}$ |
| **Product Peak** | $I[f_4] = \prod_{i=1}^{2} \left(a_i \left(arctan\,\left(a_i(1-b_i)\right) + arctan\,(a_i b_i)\right)\right.$ |
| **Corner Peak** | $I[f_5] = \dfrac{2 + a_1 + a_2}{2(1+a_1)(1+a_2)(1+a_1+a_2)}$ |
| **Gaussian** | $I[f_6] = \dfrac{\pi}{4a_2^2}\left[erf\,(a_2 b_1) - erf\,\left(a_2(b_1-1)\right)\right]\left[erf\,(a_2 b_2) - erf\,\left(a_2(b_2-1)\right)\right]$ |

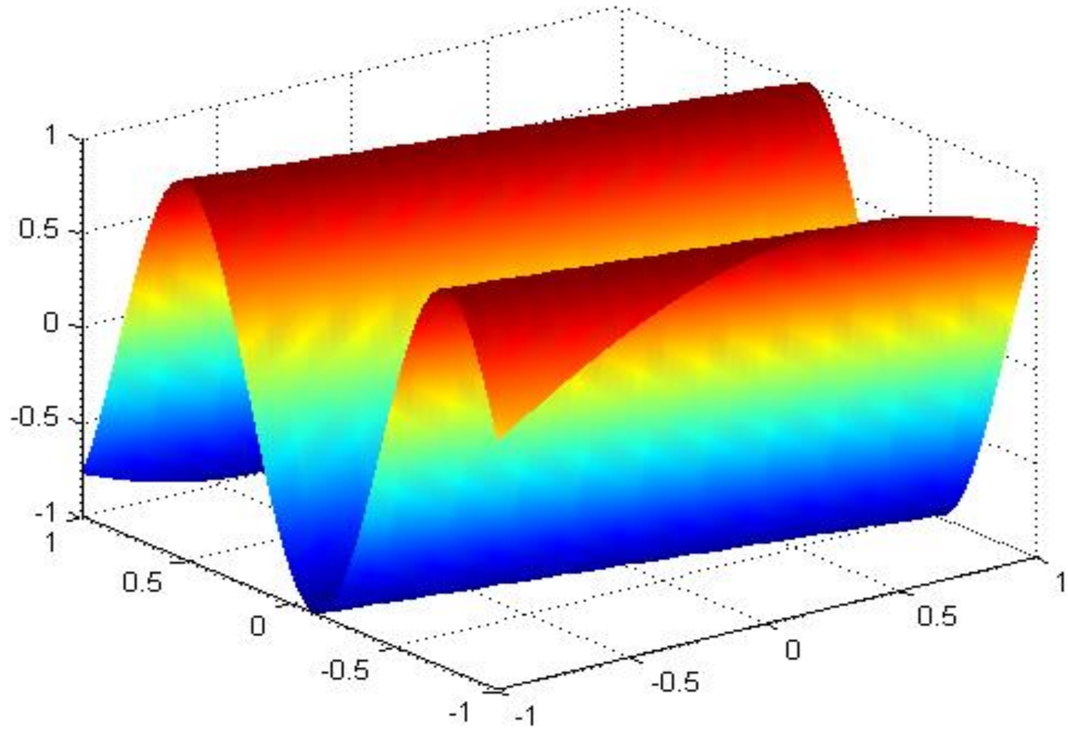


Fig. 1: Surface of $g_1$ over $\mathbb{S}_0$. $\boldsymbol{a} = (2,10)$ and $\boldsymbol{b} = (0.8,0.2)$.

1) GQS: Gaussian quadrature rule on the square $\mathbb{S}_0$ (refer to Section VII-A),

2) GQT: Gaussian quadrature rule on the triangle $\mathbb{T}_0$ (refer to Section VII-B),

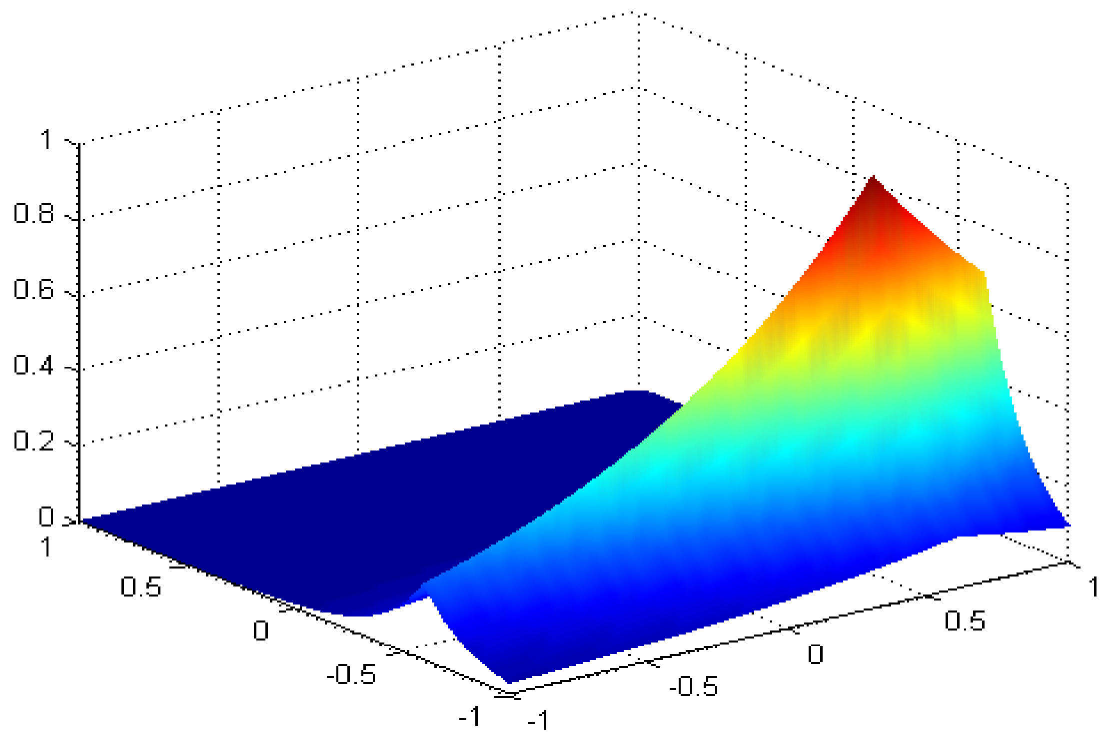


Fig. 2: Surface of $g_2$ over $\mathbb{S}_0$. $\boldsymbol{a} = (2,10)$ and $\boldsymbol{b} = (0.8,0.2)$.

3) CV-GQS1: Gaussian quadrature rule on the square $\mathbb{S}_0$ derived from GQT and the diffeomorphism whcich transforms $\mathbb{T}_0$ into $\mathbb{S}_0$ (refer to Equation (112)),

4) CV-GQS2: Gaussian quadrature rule on the square $\mathbb{S}_0$ derived from GQT and the diagonal division of $\mathbb{S}_0$ into two right triangles (refer to Definition 8.2),

5) TP-GLQS: Tensor-product Gauss-Legendre quadrature rule on the square $\mathbb{S}_0$ (refer to Equation (62) and Section III),

6) TP-GCQS: Tensor-product Gauss-Chebyshev quadrature rule on the square $\mathbb{S}_0$ (refer to Equations (62), (32) and Section III).

Table $V$ presents the number of nodes used by each method mentioned above. The nodes and their corresponding weights are collected from their original articles cited in Sections VII-A and VII-B. For a given quadrature method $Q_n$, and for a given integrand $f$, the quality of the

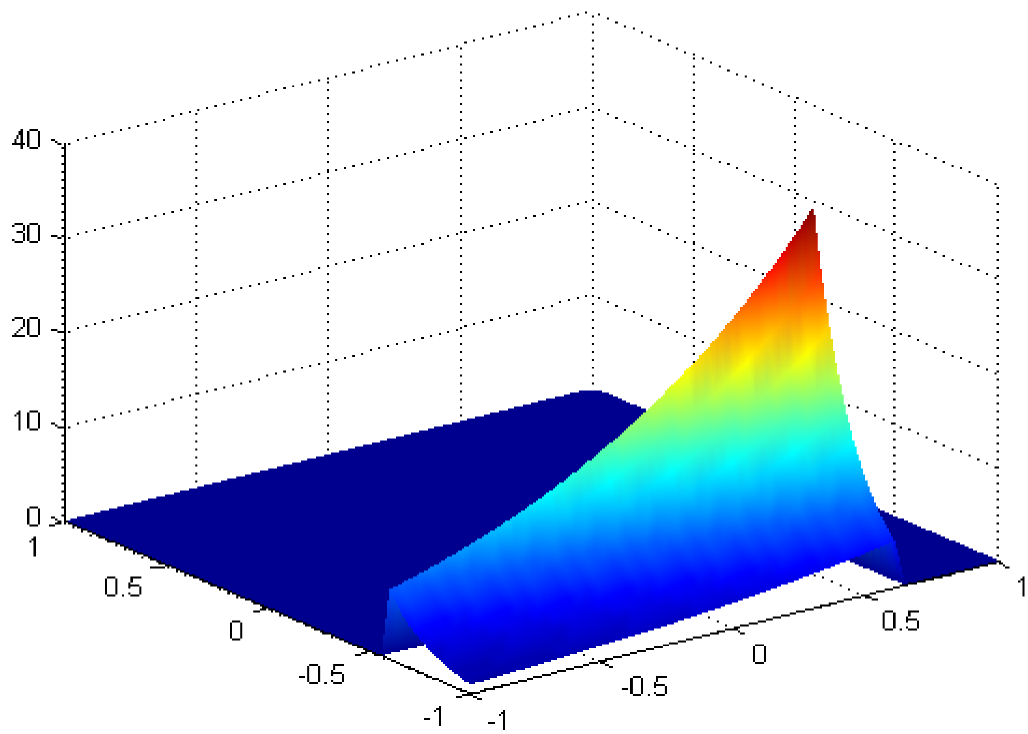


Fig. 3: Surface of $g_3$ over $\mathbb{S}_0$. $\boldsymbol{a} = (2,10)$ and $\boldsymbol{b} = (0.8,0.2)$.

**TABLE V:** Number of nodes for each quadrature method.

| **Method** | **GQS** | **TP-GLQS** | **TP-GCQS** | **GQT** | **CV-GQS1** | **CV-GQS2** |
|---|---|---|---|---|---|---|
| $\boldsymbol{n+1}$ | Table I | $[1^2, 2^2, \dots, 10^2]$ | $[1^2, 2^2, \dots, 10^2]$ | Table I | Table II | Table II |

estimate $Q_n[f]$ is measured by the percentage error $E_n$ defined as follows

$$E_n(f) = 100\frac{|Q_n[f] - I[f]|}{|I[f]|}. \quad (120)$$

C. Results

Tables VI, VII, VIII, IX and $X$ present the percentage error $E_n$ versus $n$, for GQS, CV-GQS1, CV-GQS2, TP-GLQS, TP-GCQS respectively.

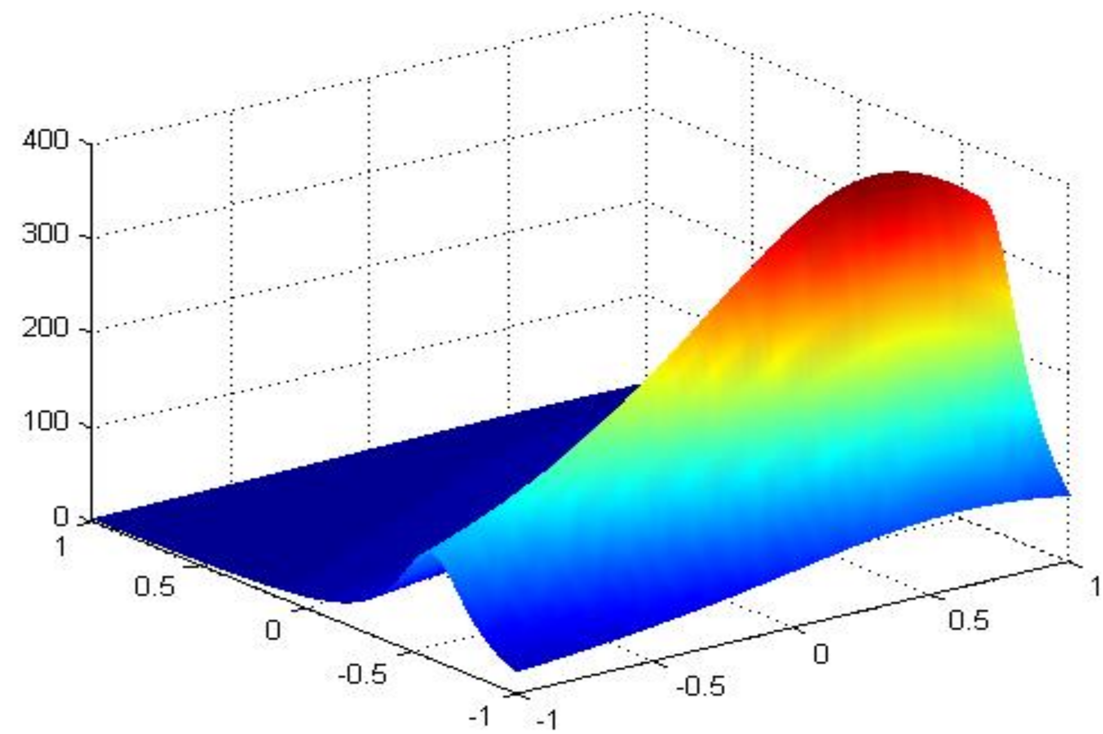


Fig. 4: Surface of $g_4$ over $\mathbb{S}_0$. $\boldsymbol{a} = (2,10)$ and $\boldsymbol{b} = (0.8,0.2)$.

D. Discussion

Based on the percentage errors shown in Tables VI, . . . , X:

- function $g_1 \in \mathcal{C}^\infty(\mathbb{S}_0)$, that is why it corresponds to the smallest percentage error. This result is consistent with the statements of Theorems 4.1, 4.2 5.1,
- we notice that the discontinuous function $g_3$ poses more difficulties for all the numerical methods. We explain the degradation of the error by the fact that the integrand cannot be well approximated on the square $[-1,1]^2$ by a unique polynomial function,
- the presence of infinity in the third and thirteenth rows of Table VII is caused by the rational fraction $\frac{1}{2-x-y}$, which tends to infinity at $(x,y) = (1,1)$,
- Although $g_6$ has an acute local peak, the GQS integration method has a relatively small percentage error. In fact, the function $g_6 \in \mathcal{C}^\infty(\mathbb{S}_0)$, so according to Theorem 4.1, the absolute error is proportional to $\frac{2M_{m+1}}{(m+1)!}$, where $m$ is the strength of GQS,

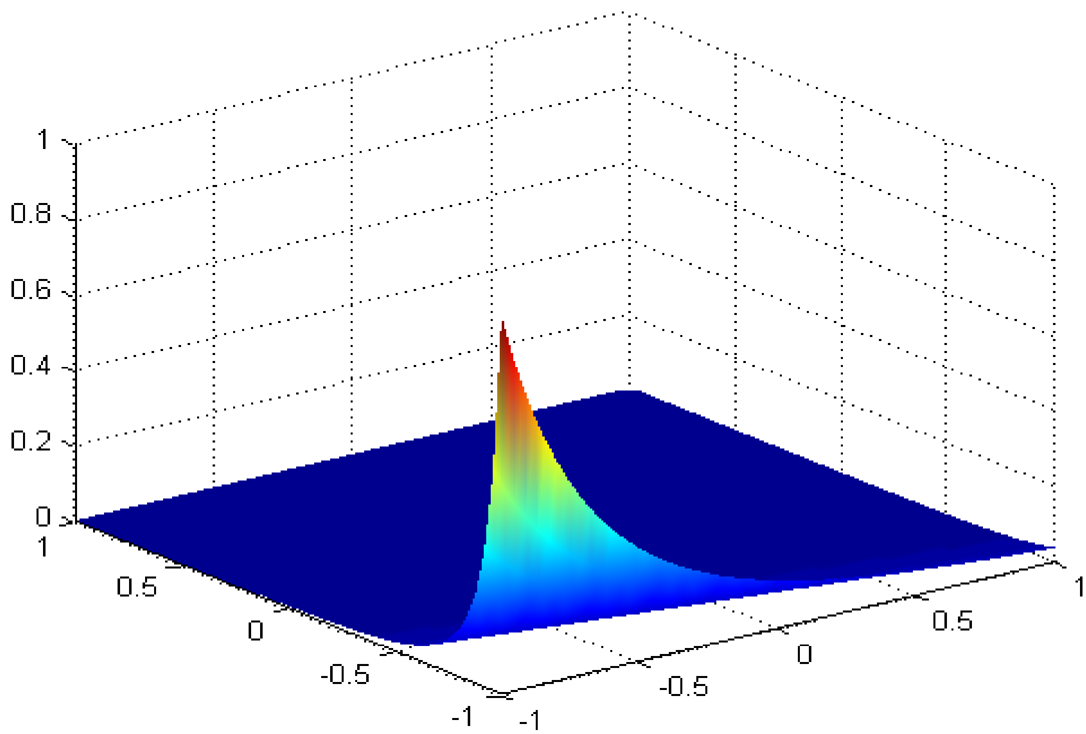


Fig. 5: Surface of $g_5$ over $\mathbb{S}_0$. $\boldsymbol{a} = (2,10)$ and $\boldsymbol{b} = (0.8,0.2)$

- although the TP-GCQS method is easier to implement, it was found to be less efficient than the TP-GLQS method.
- for a given number of nodes $n$, CV-GQS2 is more efficient than CV-GQS1,

E. Percentage error reduction

Faced with a non-smooth integrand, the percentage error of the $n$-nodes GQT method can be reduced by considering finer domains of integration. To do this, we opted for the uniform partitioning of $\mathbb{S}_0$ into 4 squares (resp. triangles) as shown in Figure 7 (resp. Figure 8). Then, we reiterate the partitioning process on each sub-square (resp. sub-triangle) to get a uniform partition of $\mathbb{S}_0$ into 16 squares (resp. triangles). So the first partition divides $\mathbb{S}_0$ into 4 , the second partition divides $\mathbb{S}_0$ into 16 , and the $d$-th partition divides $\mathbb{S}_0$ into $4^d$ squares (resp. triangles) of same area and type.

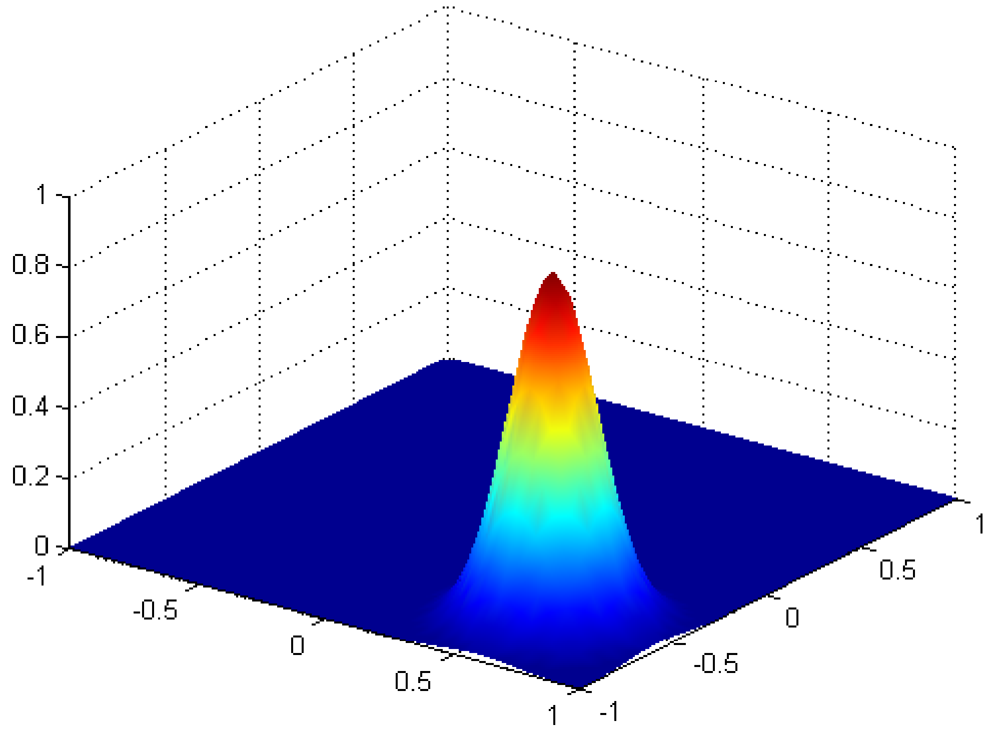


Fig. 6: Surface of $g_6$ over $\mathbb{S}_0$. $\boldsymbol{a} = (2,10)$ and $\boldsymbol{b} = (0.8,0.2)$.

Definition 8.1 (partition level): We call partition level, and we note $d$, the integer which leads to the partition of $D_0$ into $4^d$ identical squares (resp. triangles).

1) Recursive quadrature rules: In the following, we use the abbreviations introduced in Definitions 8.2 and 8.3 to designate the methods resulting from the uniform partitioning of the reference square $\mathbb{S}_0$ into sub-squares or sub-triangles

Definition 8.2 (RGQS and RGQT): We define:

1) the recursive GQS and we denote RGQS, the quadrature rule which consists in partitioning $\mathbb{S}_0$ at the level $d$ using squares, and in approximating $I[g]$ by the sum of the approximate values of the integral of $g$ on each sub-square of the partition. For a given partition level $d$, the number of nodes used by RGQS is fixed to $N_d = 85 \times 4^d$.

**TABLE VI:** Percentage error of GQS versus $n$.

| $n + 1$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| **1** | $7.20e + 2$ | $7.40e + 1$ | $1.00e + 2$ | $5.86e + 1$ | $8.21e + 1$ | $1.00$ |
| **4** | $4.02e + 2$ | $1.74e + 2$ | $1.00e + 2$ | $1.03e + 2$ | $4.73e + 1$ | $6.79$ |
| **4** | $4.02e + 2$ | $1.74e + 2$ | $1.00e + 2$ | $1.03e + 2$ | $4.73e + 1$ | $6.79$ |
| **8** | $3.11e + 1$ | $7.45$ | $1.22e + 2$ | $9.80e - 1$ | $4.70e - 1$ | $9.71e - 1$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| **8** | $3.11e+1$ | $7.45$ | $1.22e+2$ | $9.80e-1$ | $4.70e-1$ | $9.71e-1$ |
| **12** | $2.30e+1$ | $1.99e+1$ | $6.61e+1$ | $1.39e+1$ | $1.16e+1$ | $3.98e-1$ |
| **12** | $2.30e+1$ | $1.99e+1$ | $6.61e+1$ | $1.39e+1$ | $1.16e+1$ | $3.98e-1$ |
| **20** | $3.91e-1$ | $2.34$ | $3.52e+1$ | $3.51$ | $4.12e-1$ | $3.12e-1$ |
| **20** | $3.91e-1$ | $2.34$ | $3.52e+1$ | $3.51$ | $4.12e-1$ | $3.12e-1$ |
| **28** | $1.67e-1$ | $5.88$ | $3.80$ | $1.93e-1$ | $2.13$ | $7.12e-1$ |
| **28** | $1.67e-1$ | $5.88$ | $3.80$ | $1.93e-1$ | $2.13$ | $7.12e-1$ |
| **37** | $4.79e-3$ | $4.14$ | $1.06e+1$ | $1.85$ | $6.44e-1$ | $9.73e-2$ |
| **37** | $4.79e-3$ | $4.14$ | $1.06e+1$ | $1.85$ | $6.44e-1$ | $9.73e-2$ |
| **48** | $2.54e-5$ | $6.44$ | $3.76e+1$ | $1.88$ | $1.44e-2$ | $1.90e-1$ |
| **48** | $2.54e-5$ | $6.44$ | $3.76e+1$ | $1.88$ | $1.44e-2$ | $1.90e-1$ |
| **60** | $1.58e-6$ | $9.76$ | $3.90e+1$ | $1.29$ | $1.23e-2$ | $1.62e-1$ |
| **60** | $1.58e-6$ | $9.76$ | $3.90e+1$ | $1.29$ | $1.23e-2$ | $1.62e-1$ |
| **72** | $1.08e-9$ | $2.77$ | $6.67$ | $5.82e-2$ | $3.62e-3$ | $2.23e-2$ |
| **72** | $1.08e-9$ | $2.77$ | $6.67$ | $5.82e-2$ | $3.62e-3$ | $2.23e-2$ |
| **85** | $1.64e-9$ | $4.34$ | $4.73$ | $4.36e-1$ | $1.04e-2$ | $7.08e-3$ |
| **85** | $1.64e-9$ | $4.34$ | $4.73$ | $4.36e-1$ | $1.04e-2$ | $7.08e-3$ |

values of the integral of $g$ on each sub-triangle of the partition. For a given partition level $d$, the number of nodes used by RGQT is fixed to $N_d = 87 \times 4^d$.

Definition 8.3 (RTP-GCQS and RTP-GLQS): For a given partition level $d$, the number of nodes used by RGQS is $N_d = 85 \times 4^d$. We define:

1) the recursive TP-GCQS and denote RTP-GCQS, by the quadrature rule TP-GCQS using $n_d^2$ nodes, where $n_d = \lfloor \sqrt{N_d} \rfloor$, and $\lfloor \cdot \rfloor$ is the floor symbol,

2) the recursive TP-GLQS and denote RTP-GLQS, by the quadrature rule TP-GLQS using $n_d^2$ nodes.

The values of $n_d$ vary according to the partition level $d$, and are reported in Table XI. The application of TP-GLQS is limited to $d = 4$, due to the determination of the Legendre polynomial

**TABLE VII:** Percentage error of CV-GQS1 versus n.

| $n+1$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| **1** | $1.34e+4$ | $5.82e+1$ | $1.00e+2$ | $4.79e+1$ | $7.63e+1$ | $1.00e+2$ |
| **3** | $8.53e+2$ | $1.02e+2$ | $4.62e+1$ | $7.65e+1$ | $4.45e+1$ | $5.17e+2$ |
| **6** | $Inf$ | $Inf$ | NAN | Inf | Inf | Inf |
| **6** | $4.51e+3$ | $2.06e+1$ | $5.44e+1$ | $2.25e+1$ | $2.10e+1$ | $4.57e+1$ |
| **7** | $2.98e+3$ | $2.17e+1$ | $3.68e+1$ | $1.20e+1$ | $1.81e+1$ | $1.29e+2$ |
| **12** | $1.64e+3$ | $2.80e+1$ | $3.73e+1$ | $2.03e+1$ | $7.81$ | $7.50e+1$ |
| **15** | $1.88e+3$ | $1.92e+1$ | $5.27e+1$ | $1.15e+1$ | $6.99$ | $7.38e+1$ |
| **16** | $9.88e+2$ | $2.14$ | $3.00$ | $2.20$ | $8.78e-1$ | $7.58e+1$ |
| **19** | $6.72e+2$ | $6.70$ | $1.74$ | $4.70$ | $2.48$ | $1.81e+1$ |
| **25** | $2.00e+2$ | $8.14$ | $1.01e+1$ | $5.29$ | $6.37e-1$ | $1.90$ |
| **28** | $9.01e+4$ | $2.54e+3$ | $1.70e+3$ | $2.56e+3$ | $3.77e+3$ | $3.07e+3$ |
| **33** | $8.91$ | $7.34$ | $3.35e+1$ | $4.94$ | $3.86e-1$ | $2.54e+1$ |
| **37** | $Inf$ | $Inf$ | $NAN$ | $Inf$ | $Inf$ | $Inf$ |
| **42** | $3.00e+1$ | $5.37$ | $1.08e+1$ | $3.65$ | $7.90e-3$ | $6.88$ |
| **49** | $2.14e+1$ | $5.11$ | $7.28$ | $3.12$ | $8.53e-3$ | $4.91$ |
| **55** | $1.34e-1$ | $4.10$ | $5.69$ | $1.72$ | $3.86e-2$ | $9.86e-2$ |
| **60** | $6.74e+1$ | $2.21$ | $4.10$ | $6.13e-1$ | $1.29e-2$ | $6.64$ |
| **67** | $4.10e+1$ | $1.51$ | $1.04e+1$ | $2.67e-1$ | $2.64e-2$ | $9.17e-1$ |
| **73** | $3.49e-3$ | $1.19$ | $2.43e+1$ | $2.05e-1$ | $1.98e-2$ | $3.16e-1$ |
| **79** | $3.21e-3$ | $4.26$ | $6.38$ | $5.77e-1$ | $1.52e-2$ | $7.02e-1$ |
| **87** | $2.98e-3$ | $9.66e-1$ | $1.33e+1$ | $4.30e-1$ | $9.82e-3$ | $1.54e-1$ |

roots of degree $n_d$. The application of TP-GCQS has no constraints since the roots of the Chebyshev polynomials are explicit.

2) Results: Tables XII, XIII, XIV and XV present the percent error $E_{N_d}$ versus $d$, for RGQS, RGQT, RTP-GLQS, RTP-GCQS respectively.

3) Discussion:

- From partition level $d = 1$, RGQS and RGQT becomes more efficient than all other methods.
- With the exception of the integrand $g_3$, we observe a slow and monotonous convergence of TP-GCQS.
- Although TP-GLGQ is more efficient for small and medium values of $n$, it has been found to diverge for large values of $n$. This is explained by the difficulty of calculating the zeros

**TABLE VIII:** Percentage error of CV-GQS2 versus n.

| $n+1$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| **2** | $1.58e+2$ | $4.73e+1$ | $1.00e+2$ | $3.44e+1$ | $7.13e+1$ | $1.00e+2$ |
| **6** | $1.28e+2$ | $5.89e+1$ | $2.63e+2$ | $4.98e+1$ | $3.77e+1$ | $6.12e+1$ |
| **12** | $6.74e+1$ | $4.57e+1$ | $2.30e+2$ | $3.88e+1$ | $2.98e+1$ | $4.51e+1$ |
| **12** | $5.64e+1$ | $1.08e+1$ | $2.33e+1$ | $1.04e+1$ | $1.61e+1$ | 5.49 |
| **14** | $1.54e+1$ | 1.89 | 8.86 | 6.74 | $1.50e+1$ | $5.19e+1$ |
| **24** | 3.50 | $1.59e+1$ | $6.54e+1$ | 8.53 | 5.97 | $7.80e+1$ |
| **30** | 1.73 | 6.32 | $2.13e+1$ | 5.71 | 5.64 | $7.64e+1$ |
| **32** | $3.63e-1$ | 3.69 | $4.63e+1$ | 1.52 | 1.25 | $1.81e+1$ |
| **38** | $1.06e-1$ | 8.73 | $4.88e+1$ | 5.08 | 2.06 | $1.35e+1$ |
| **50** | $3.83e-2$ | 1.48 | $4.79e+1$ | $6.28e-1$ | $2.30e-1$ | $3.55e+1$ |
| **56** | $7.86e-3$ | 2.00 | $1.59e+1$ | $2.11e-1$ | $6.34e-1$ | $2.16e+1$ |
| **66** | $9.51e-4$ | $1.03e+1$ | $1.83e+1$ | 4.16 | $2.44e-1$ | $1.28e+1$ |
| **74** | $1.37e-4$ | 4.72 | $2.09e+1$ | 1.13 | $2.37e-1$ | $1.55e+1$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| **84** | $2.04e-5$ | $8.87e-1$ | $3.62e+1$ | $3.23e-2$ | $1.46e-2$ | $2.21$ |
| **98** | $1.92e-6$ | $2.82$ | $1.33e+1$ | $7.41e-1$ | $4.43e-2$ | $1.32$ |
| **110** | $3.15e-7$ | $1.20$ | $2.25e+1$ | $5.31e-1$ | $2.50e-2$ | $5.26$ |
| **120** | $2.22e-9$ | $2.07$ | $1.52e+1$ | $3.49e-1$ | $1.27e-2$ | $3.07$ |
| **134** | $1.32e-8$ | $3.97$ | $1.54e+1$ | $7.48e-1$ | $3.94e-3$ | $1.55$ |
| **146** | $5.98e-10$ | $7.23e-1$ | $6.73$ | $1.07e-1$ | $4.21e-3$ | $2.86$ |
| **158** | $7.12e-11$ | $7.52e-1$ | $7.08$ | $1.14e-1$ | $4.44e-4$ | $5.63e-1$ |
| **174** | $1.45e-11$ | $1.11e-1$ | $1.13$ | $1.17e-1$ | $2.58e-3$ | $1.13$ |

of the Legendre polynomials for large values of $n$

- The RGQS and RQGT methods gave almost equal relative errors. Moreover, applied to integrands $g_1$ and $g_2$, they show a relatively slow convergence speed. This again confirms that quadrature methods are sensitive to non-smooth integrands.
- Moreover, partitioning the integration domain $D$ into sub-squares or sub-triangles has no significant effect on the relative error. However, if $D$ is not rectangular, then triangulation is the technique which minimizes the error of approximating $D$ by a finite union of elementary sub-domains.

**TABLE IX:** Percentage error of TP-GLQS versus n.

| **n+ 1** | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| **1** | $7.20e+2$ | $7.40e+1$ | $1.00e+2$ | $5.86e+1$ | $8.21e+1$ | $1.00e+2$ |
| **4** | $4.02e+2$ | $1.74e+2$ | $1.00e+2$ | $1.03e+2$ | $4.73e+1$ | $6.79e+2$ |
| **9** | $1.16e+2$ | $2.79e+1$ | $5.65$ | $1.91e+1$ | $2.13e+1$ | $4.62e+1$ |
| **16** | $1.55e+1$ | $2.68e+1$ | $4.27e+1$ | $2.05e+1$ | $8.41$ | $7.31e+1$ |
| **25** | $1.21$ | $1.77e+1$ | $6.40e+1$ | $1.42e+1$ | $3.06$ | $6.79e+1$ |
| **36** | $6.25e-2$ | $2.83$ | $4.42e+1$ | $2.21$ | $1.07$ | $2.00$ |
| **49** | $2.28e-3$ | $1.26e+1$ | $8.85$ | $6.12$ | $3.59e-1$ | $1.58e+1$ |
| **64** | $6.24e-5$ | $8.46e-1$ | $3.67e+1$ | $1.48$ | $1.18e-1$ | $7.30$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| **81** | $1.32e-6$ | $1.10e+1$ | $5.37e+1$ | $1.77$ | $3.83e-2$ | $6.61e-2$ |
| **100** | $2.24e-8$ | $5.30$ | $8.78$ | $1.27$ | $1.23e-2$ | $1.32$ |

**TABLE X:** Percentage error of TP-GCQS versus $n$.

| $\boldsymbol{n+1}$ | $\boldsymbol{g_1}$ | $\boldsymbol{g_2}$ | $\boldsymbol{g_3}$ | $\boldsymbol{g_4}$ | $\boldsymbol{g_5}$ | $\boldsymbol{g_6}$ |
|---|---|---|---|---|---|---|
| **1** | $1.63e+3$ | $3.59e+1$ | $1.00e+2$ | $2.03$ | $5.59e+1$ | $1.00e+2$ |
| **4** | $4.37e+2$ | $1.01e+2$ | $4.66e+2$ | $8.84e+1$ | $1.95e+1$ | $4.56e+2$ |
| **9** | $2.76e+2$ | $5.02e+1$ | $3.91e+2$ | $3.92e+1$ | $3.66e+1$ | $9.36e+1$ |
| **16** | $4.38e+1$ | $1.61e+1$ | $3.94e+2$ | $9.94$ | $2.91e+1$ | $5.21e+1$ |
| **25** | $7.61$ | $5.28e+1$ | $4.08e+2$ | $2.58e+1$ | $1.98e+1$ | $1.12e+2$ |
| **36** | $1.35$ | $9.31$ | $1.33e+3$ | $4.18$ | $1.32e+1$ | $2.30e+1$ |
| **49** | $1.10$ | $1.19e+1$ | $1.16e+3$ | $5.10$ | $9.12$ | $1.38e+1$ |
| **64** | $7.65e-1$ | $8.58$ | $1.07e+3$ | $5.17$ | $6.59$ | $1.33e+1$ |
| **81** | $5.68e-1$ | $3.07$ | $2.26e+3$ | $1.20$ | $4.98$ | $2.70$ |
| **100** | $4.38e-1$ | $6.82$ | $1.98e+3$ | $1.35$ | $3.91$ | $1.16$ |

**TABLE XI:** Values of $n_d$ according to the partition level $d$.

| $\boldsymbol{d}$ | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| $\lfloor \boldsymbol{n_d} \rfloor$ | 19 | 37 | 74 | 148 | 296 | 591 |

**IX. CONCLUSION**

1) In this article, we give an overview of the different PIS quadrature rules for triangle and square up to strength 21.

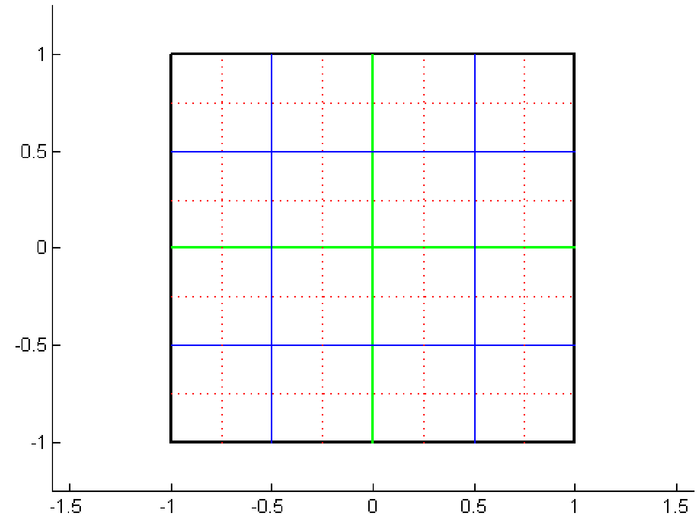


Fig. 7: The subdivision of $\mathbb{S}_0$ using the green lines corresponds to the partition level d = 1.

The subdivision using the green and blue lines corresponds to d = 2. The subdivision using the green, blue and red lines corresponds to d = 3.

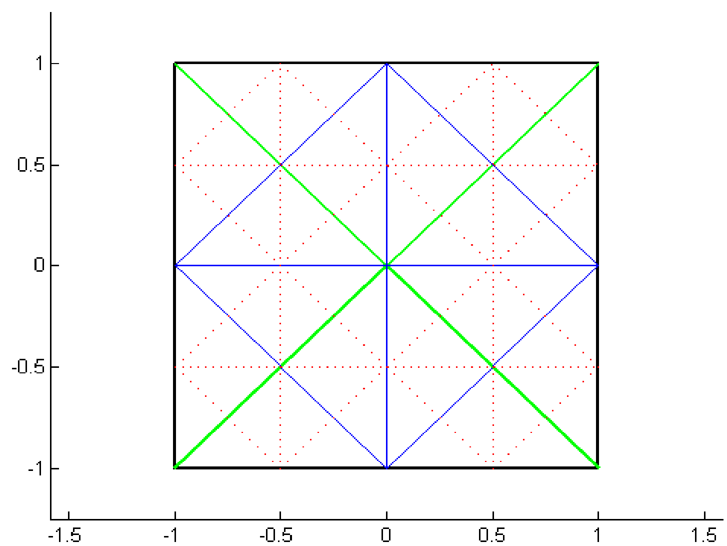


Fig. 8: The subdivision of $\mathbb{S}_0$ using the green lines corresponds to the partition level d = 1.

The subdivision using the green and blue lines corresponds to d = 2. The subdivision using the green, blue and red lines corresponds to d = 3.

2) For regular integrands, GQS and GOT are most relevant for numerical integration over squares and triangles respectively.

3) Quadrature rules are sensitive to irregular integrands since this class of functions cannot be

**TABLE XII:** Percentage error of RGQS versus $n$.

| $d$ | $g_2$ | $g3$ | $g_4$ | $g_5$ | $g6$ |
|---|---|---|---|---|---|
| **1** | $9.50e-4$ | $1.12e-1$ | $2.71e-5$ | $5.96e-7$ | $3.44e-6$ |
| **2** | $2.77e-3$ | $1.53e-2$ | $8.87e-11$ | $5.13e-10$ | $1.50e-10$ |
| **3** | $6.08e-5$ | $4.15e-2$ | $1.53e-13$ | $5.12e-14$ | $0.00$ |
| **4** | $1.73e-4$ | $1.86e-3$ | $0.00$ | $0.00$ | $0.00$ |
| **5** | $3.80e-6$ | $9.63e-3$ | $0.00$ | $0.00$ | $0.00$ |
| **6** | $1.08e-5$ | $5.79e-4$ | $0.00$ | $0.00$ | $0.00$ |

**TABLE XIII:** Percentage error of RGQT versus $n$.

| $d$ | $g2$ | $g3$ | $g_4$ | $g_s$ | $g6$ |
|---|---|---|---|---|---|
| **1** | $7.77e-3$ | $4.93e-2$ | $4.37e-4$ | $6.35e-6$ | $4.28e-6$ |
| **2** | $1.22e-3$ | $2.02e-3$ | $4.16e-6$ | $2.52e-8$ | $3.33e-9$ |
| **3** | $8.28e-4$ | $1.45e-2$ | $6.31e-8$ | $1.44e-11$ | $1.36e-12$ |
| **4** | $3.88e-5$ | $7.16e-3$ | $5.87e-11$ | $8.51e-16$ | $1.33e-15$ |
| **5** | $5.39e-5$ | $2.97e-3$ | $1.62e-13$ | $0.00$ | $0.00$ |
| **6** | $7.50e-7$ | $1.55e-4$ | $9.99e-16$ | $0.00$ | $0.00$ |

**TABLE XIV:** Percentage error of RTP-GLQS versus $n$.

| $d$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g6$ |
|---|---|---|---|---|---|---|
| **1** | $3.95e-8$ | $8.98e-1$ | $1.43e+1$ | $1.53e-2$ | $1.05e-6$ | $2.44e-4$ |
| **2** | $3.47e-2$ | $9.82e-1$ | $1.52e+1$ | $1.02e-2$ | $6.88e-2$ | $9.38e-4$ |
| **3** | $1.15e+2$ | $9.92e+1$ | $1.00e+2$ | $9.88e+1$ | $9.95e+1$ | $1.00e+2$ |
| **4** | $9.88e+1$ | $9.99e+1$ | $1.00e+2$ | $9.98e+1$ | $9.99e+1$ | $1.00e+2$ |

approximated on the square or the triangle by a single polynomial.

4) Uniform subdivision of the integration domain using squares or triangles reduces the percentage

error from the partition level d = 1.

5) Although TP-GLGQ is more efficient for small and medium values of n, it was found to diverge for large values of n.

6) Except for discontinuous integrands, TP-GCQS converges slowly and monotonously to the

**TABLE XV:** Percentage error of RTP-GCQS versus $n$.

| $d$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| **1** | $1.15e-1$ | $7.11e-1$ | $3.24e+3$ | $1.92e-1$ | $1.11$ | $1.21e-2$ |
| **2** | $2.71e-2$ | $4.10e-1$ | $8.12e+3$ | $3.94e-2$ | $2.72e-1$ | $3.24e-3$ |
| **3** | $6.49e-3$ | $2.14e-1$ | $1.61e+4$ | $9.59e-3$ | $6.58e-2$ | $7.97e-4$ |
| **4** | $1.59e-3$ | $3.10e-2$ | $3.06e+4$ | $2.36e-3$ | $1.62e-2$ | $1.97e-4$ |

exact value of the integral.

7) Constructing new quadrature rules is still possible thanks to the continuous growth of supercomputers.

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## CONFLICT OF INTEREST

The author declares that there is no conflict of interest.

## DATA AVAILABILITY

The raw data required to reproduce the above findings are available to download from [https://github.com/abdelhamidzaidi/Quadrature-nodes-and-weights-for-the](https://github.com/abdelhamidzaidi/Quadrature-nodes-and-weights-for-the) square-and-the triangle.

## REFERENCES

[1]. time derivative of bioheat model in skin tissue induced to laser irradiation," Symmetry, vol. 12, no. 4, 20 E. Lindgren et al, "An integral equation approach to calculate electrostatic interactions in many-body dielectric systems," Journal of Computational Physics, vol. 371, pp. 712-731, 2018.

[2]. D. Baskaran et al, "Center of mass integral in canonical general relativity," Annals of Physics, vol. 307, no. 1, pp. 90-131, 2003.

[3]. A. Hobiny et al, "The effect of fractional 20.

[4]. M. Marin et al, "An extension of the domain of influence theorem for generalized thermoelasticity of anisotropic material with voids," Journal of Computational and Theoretical Nanoscience, vol. 12, no. 8, pp. 1594-1598, 2015.

[5]. J. Grammer and K. Kinoshita, "The two photon process for particle production in colliding beam experiments," Nucl. Phys. B, vol. 80, pp. $461-490$,1974.

[6]. G. Peter Lepage, "A new algorithm for adaptive multidimensional integration," Journal of Computational Physics, vol. 27, no. 2, pp. 192–203, 1978.

[7]. A. Becke, "A multicenter numerical integration scheme for polyatomic molecules," The Journal of Chemical Physics, vol. 88, no. 4 , pp. 2547 $-2553$,1988.

[8]. S. Blinder and J. House, "Chapter 2 - slater and gaussian basis functions and computation of molecular integrals," pp. 31-61, 2019.

[9]. O. Nechuiviter et al, "Numerical calculation of multidimensional integrals depended on input information about the function in mathematical modelling of technical and economic processes,"

Conference Series: Materials Science and Engineering, vol. 1031, no. 1, p. 012059,2021.

[10]. A. Papageorgiou and J. Traub, "Faster evaluation of multidimensional integrals," Computers in Physics, vol. 11, no. 6, pp. 574 − 579,1997.

[11]. A. Dixit, Optimization in economic theory. Oxford University Press, 1990.

[12]. M. Intriligator, Mathematical Optimization and Economic Theory, 1st ed. SIAM, 2012.

[13]. A. Mazzia and G. Pini, "Product gauss quadrature rules vs. cubature rules in the meshless local petrov-galerkin method," Journal of Complexity, vol. 26, no. 1, pp. 82-101, 2010.

[14]. A. Veilleux, "Extension of the Spectral Difference method to simplex cells and hybrid grids," Theses, UNIVERSITE DE TOULOUSE, 2021. [Online]. Available: https://hal. archives-ouvertes.fr/tel-03299370.

[15]. M. Evans and T. Swartz, "Methods for approximating integrals in statistics with special emphasis on bayesian integration problems," Statistical Science, vol. 10, no. 3, pp. 254-272, 1995.

[16]. A. Brian et al, "Symmetric triangle quadrature rules for arbitrary functions," Computers and Mathematics with Applications, vol. 79, no. 10, pp. 2885-2896, 2020.

[17]. M. Festa and A. Sommariva, "Computing almost minimal formulas on the square," Journal of Computational and Applied Mathematics, vol. 236, no. 17, pp. 4296-4302, 2012.

[18]. K. Kim and M. Song, "Symmetric quadrature formulas over a unit disk," Journal of applied mathematics and informatics, no. 1,1997.[19]

[19]. S. Mousavi et al, "Generalized gaussian quadrature rules on arbitrary polygons," International Journal for Numerical Methods in Engineering, vol. 82, no. 1, pp. 99-113.

[20]. F. Witherden and P. Vincent, "On the identification of symmetric quadrature rules for finite element methods," Computers & Mathematics with Applications, vol. 69, no. 10, pp. 1232–1241, 2015.

[21]. H. Xiao and Z. Gimbutas, “A numerical algorithm for the construction of efficient quadrature rules in two and higher dimensions,” Computers and mathematics with applications, vol. 59, no. 2, pp. 663–676, 2010

[22]. D. Marquardt, “An algorithm for least-squares estimation of nonlinear parameters,” Journal of the Society for Industrial and Applied Mathematics, vol. 11, no. 2, pp. 431–441, 1963.

[23]. J. Lyness and D. Jespersen, “Moderate degree symmetric quadrature rules for the triangle,” IMA Journal of Applied Mathematics, vol. 15, no. 1, pp. 19–32, 1975.

[24]. V. Krylov and A. Stroud, Approximate Calculation of Integrals, 1st ed. Dover Publications, 2006.

[25]. M. Abramowitz and I. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables. US Government printing office, 1964, vol. 55.

[26]. J. Duistermaat, , and J. A. C. Kolk, Taylor Expansion in Several Variables. Birkh¨auser Boston, 2010.

[27]. W. Burnside, “An approximate quadrature formula,” The Messenger of Mathematics, vol. 37, pp. 166–167, 1908.

[28]. G. Tyler, “Numerical integration of functions of several variables,” Canadian Journal of Mathematics, vol. 5, pp. 393–412, 1953.

[29]. P. Rabinowitz and N. Richter, “Perfectly symmetric two-dimensional integration formulas with minimal numbers of points,”Mathematics of Computation, vol. 23, no. 108, pp. 765–779, 1969.

[30]. D. Dunavant, “Economical symmetrical quadrature rules for complete polynomials over a square domain,” International journal for numerical methods in engineering, vol. 21, no. 10, pp. 1777–1784, 1985.

[31]. H. Gavin, “The levenberg-marquardt algorithm for nonlinear least squares curve-fitting problems,” Department of Civil and Environmental Engineering, Duke University, vol. 19, 2019.

[32]. P. Hammer and A. Stroud, “Numerical integration over simplexes,” Mathematical tables and other aids to computation, vol. 10, no. 55, pp. 137–139, 1956.

[33]. J. Albrecht and L. Collatz, “Zur numerischen auswertung mehrdimensionaler integrale,” ZAMM-Journal of Applied Mathematics and Mechanics, vol. 38, no. 1-2, pp. 1–15, 1958.

[34]. G. Cowper, “Gaussian quadrature formulas for triangles,” International Journal for Numerical Methods in Engineering, vol. 7, no. 3, pp. 405–408, 1973.

[35]. J. Radon, “Zur mechanischen kubatur,” Monatshefte f¨ur Mathematik, vol. 52, no. 4, pp. 286–300, 1948.

[36]. M. Laursen and M. Gellert, “Some criteria for numerically integrated matrices and quadrature formulas for triangles,” International journal for numerical methods in engineering, vol. 12, no. 1, pp. 67–76, 1978.

[37]. T. C. L. Zhang and H. Liu, “A set of symmetric quadrature rules on triangles and tetrahedra,” Journal of Computational Mathematics, pp. 89–96, 2009.

[38]. J. More et al, “User guide for minpack-1. [in fortran],” 8 1980. [Online]. Available: https://www.osti.gov/biblio/6997568

[39]. D. Dunavant, “High degree efficient symmetrical gaussian quadrature rules for the triangle,” International journal for numerical methods in engineering, vol. 21, no. 6, pp. 1129–1148, 1985.

[40]. J. Berntsen and T. Espelid, Degree 13 symmetric quadrature rules for the triangle. University of Bergen. Department of Informatics, 1990.

[41]. A. Genz, "Testing multidimensional integration routines," Proceedings of International Conference on Tools, Methods and

[42]. Languages for Scientific and Engineering Computation, pp. 81–94, 1984.

Abdelhamid T. ZAIDI. is an Assistant Professor in the College of Science at Qassim University in Saudi Arabia. He has a PhD in Statistics and Stochastic Modeling from University Grenoble-Alpes (France) and an Engineering degree in Computer Science and Applied Mathematics from ENSIMAG Grenoble (France). He works mainly on the development of computational methods applied to various subjects of

signal and image processing. His research work was published in many top ranked journals. He is also the author of three books covering numerical analysis, algorithmic, probability, and statistics.