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COMPARISON BETWEEN ANALYTICAL AND NUMERICAL METHODS IN THE DETERMINATION OF THE DEFLECTION OF A SOLID PLATE

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Abstract

In the last decades, different methods have been developed for the analysis and design of structures. The objective of this work is to compare the effectiveness of the numerical methods with respect to the analytical method. For this, the deflection of a solid slab was determined, first with a uniformly distributed load, and then, with a uniform load plus an axial compression load, by the finite difference and finite element methods (STAAD-PRO). , compared to the analytical solution. The results show the comparison of the three methods, through tables and graphs for each of the load cases, in which it can be observed that the difference between the three is very small, when an adequate discretization is used.

Keywords: structures analysis, analytical methods, numerical methods, deflection, solid slab

1. INTRODUCTION

Slabs are used in most of the world's buildings to achieve flat and usable surfaces (Designing Buildings, April 2022). These slabs are subjected to loads that cause deflections and deformations that lead to subsequent stresses, which can be detrimental to the structure (Galyautdinov, 2017). These deflections can be determined by methods of elastic and plastic theory (Andreaus and Sawczuk, 1983), (Muspratt, 1978) and (Hillerborg, 1960). In the present work, the solution of the plate deformation problem is proposed using three methods: analytical (double integration method), approximate numerical, by the method of centered finite differences, and numerical, by the finite element method using Staad-Pro as a computer tool for the calculation of structures.

2. APPROACH TO THE PROBLEM

Structures, like all other physical bodies, deform and change when subjected to service loads (Designing Buildings, November 2022). For most structures, excessive deformations are undesirable, as they impair the structure's ability to serve its intended purpose. Structures are usually designed so that their deflections, in service conditions, do not exceed the allowable values specified in building codes (Constro, 2021).

The problem that arises below is that deflections and deformations not only occur due to uniform loads, but also due to axial compression loads. Then, the simply supported solid plate shown in figure 1 is considered, with a free span of 2.00 m, which is supported on two load-bearing walls. The plate is 24.5 MPa (3500 PSI) concrete with a Poisson's ratio of 0.20. The thickness of the plate is 10 cm, and the area is 4.00 m2. The load rating of the plate gave a distributed load of 10 KN/m2. The plate will also be analyzed with a compression load of 10 KN. Initially, the plate was tested with only a uniformly distributed load, and later, it was tested with the uniformly distributed load and a compression load.

CHARACTERISTICS OF THE SLAB:

Plate thickness of 10 cm

Concrete of 24.5 MPa of compressive strength

E = 20,000,000 KN/m2

Poisson's ratio = 0.20

Free span = 2.00 m

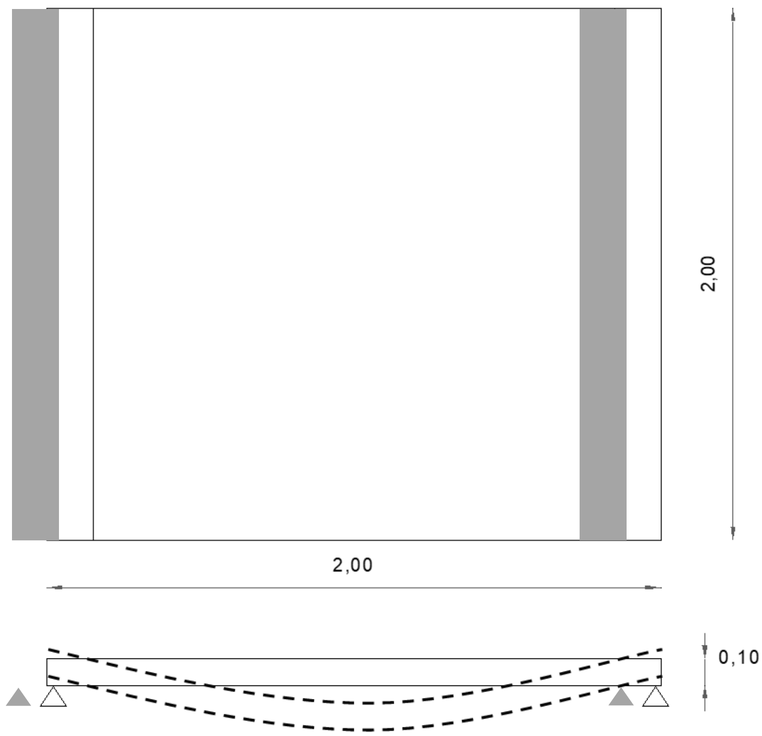

Figure 1. Characteristics of the solid plate

The procedure to solve this problem begins with the determination of the moments in each one of the points of the grid in which the length of the plate is divided. Then the deflection of each of these points is determined, applying the different methods, which are Finite difference using centered differences, Lugo the analytical method and finally the Staad-Pro.

3. SOLUTION OF THE DEFLECTION PROBLEM

According to Bedfor (2002), when considering the deflection in x and in x + Δx (figure 2), it is observed that ω and θ are related by the following equation:

$$\frac{d\omega}{dx} = \tan(\theta) = \theta + \frac{1}{3}\theta^3 + ..., \qquad (1)$$

where tan (θ) is expressed in terms of a Taylor series. We restrict our analysis to beams and loads for which θ is small enough so that second-order and higher-order terms can be ignored, so that

$$\frac{d\omega}{dx} = \theta \qquad (2)$$

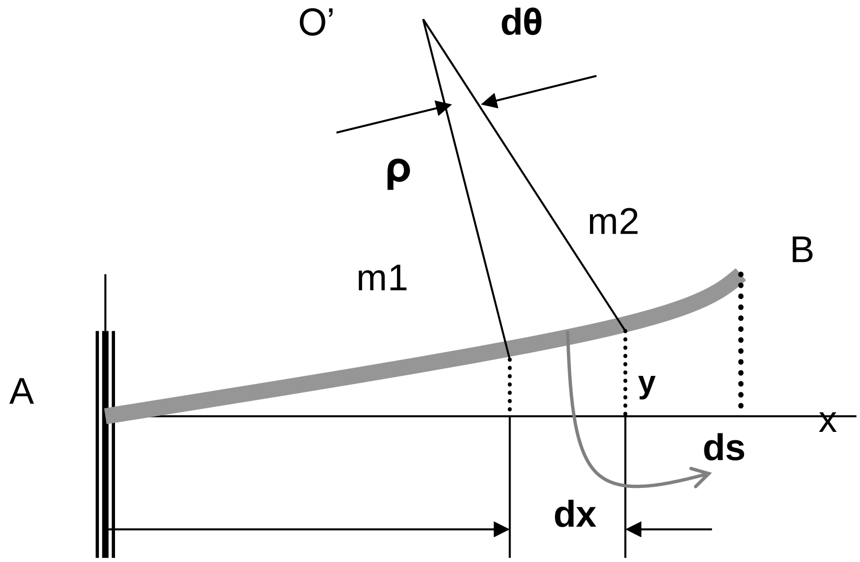

Figure 2. Deflection "y", and the angle dθ.

In Figure 3, lines are drawn perpendicular to the neutral axis at x and at x + dx. The angle dθ between these lines, is equal to the change of θ from x to x + dx. In terms of the radius of curvature of the neutral axis (ρ) and l distance ds, the angle dθ is:

$$d\theta = \frac{1}{\rho} ds \quad dx = ds\cos\theta = ds(1 - \frac{1}{2} d\theta^2 + \ldots) \quad (3)$$

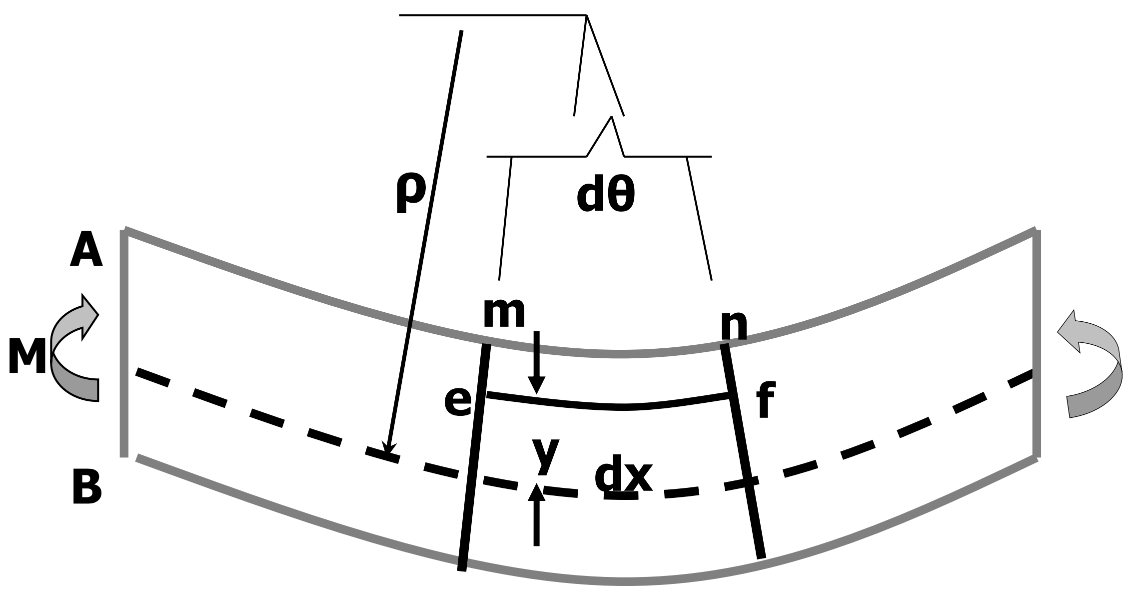

Figure 3. Relationship between θ and the radius of gyration ρ.

Consequently, ds = dx when the second order terms are not taken into account, and of higher order in θ, for which the expression can be represented as:

$$\frac{d\theta}{dx} = \frac{1}{\rho} \qquad (4)$$

Substituting equation (2) into this expression, we get

$$\frac{d^2\omega}{dx^2} = \frac{1}{\rho} \qquad (5)$$

If the material of a beam is linearly elastic and obeys Hooke's law, the curvature k is:

$$k = \frac{1}{\rho} = \frac{M}{EI} \qquad (6)$$

where M is the bending moment and EI is the bending stiffness of the beam. Equation 6 shows that a positive bending moment produces positive curvature and a negative bending moment produces negative curvature (Gere, 2002).

By combining equation 6 with equation 5, the differential equation of the basic deflection curve of a beam results:

$$\frac{d^2\omega}{dx^2} = \frac{M}{EI} \qquad (7)$$

4. NUMERICAL METHOD: FINITE DIFFERENCES BY CENTERED DIFFERENCES

The equation that governs this phenomenon or process is:

$$D\frac{d^2w}{dx^2} = -M_{(x)} \qquad (8)$$

$$D = \frac{Eh^3}{12(1-v^2)} \qquad (9)$$

Where,

h : Plate thickness

E : Elastic modulus of the material

v : Poisson's ratio

And

$$M_{(x)} = \frac{ql}{2}x - \frac{qx^2}{2} - sw \qquad (10)$$

Where,

l : Length or width of the plate

q : Uniformly distributed load

s : Axial load on the x-axis

The analytical solution of this equation is:

$$w = \frac{ql^4}{16u^4}\left[\frac{\cosh u\left(1-\frac{2x}{l}\right)}{\cosh u} - 1\right] + \frac{ql^2x}{8u^2D}(l-x)$$

(11)

Where,

$$u = \sqrt{\frac{Sl^2}{4D}} \qquad (12)$$

For the numerical solution of this equation, Centered Differences is used in the term on the left.

$$\frac{d^2w}{dx^2} = \frac{w_{(x+\Delta x)} + w_{(x-\Delta x)} - 2w_{(x)}}{(\Delta x)^2} = \frac{-M_{(x)}}{D}$$

(13)

$$\Rightarrow \frac{w_{(x+\Delta x)} + w_{(x-\Delta x)} - 2w_{(x)}}{(\Delta x)^2} = \frac{-M_{(x)}}{D}$$

$$w_{(x+\Delta x)} = 2w_{(x)} - w_{(x-\Delta x)} - \frac{(\Delta x)^2 M_{(x)}}{D}$$

$$\Rightarrow w_{(i+1)} = 2w_{(i)} - w_{(i-1)} - \frac{(\Delta x)^2 M_{(x)}}{D}$$

(14)

For the simplest case of equation 14, which corresponds to that of a rectangular beam where the term $D = EI$, being I the moment of inertia of the rectangular cross section, equation 14 looks like this:

$$w_{(i+1)} = 2w_{(i)} - w_{(i-1)} - \frac{(\Delta x)^2 M_{(x)}}{EI} \qquad (15)$$

From this equation, a program was written in *Fortran77* that calculates the value of the deformation of a beam with increments = 0.10 m. The program code is shown below, together with the input data and conditions.

Input data:

W at the origin = 0

W at the opposite end = 0

q = 1000 kgf/m

Length of the beam = 2.00 m

E = 2000000 kgf/cm2

I = 1000 cm4

Fortran Code:

```
c This program corresponds to the numerical solution of the equation
c for deformation of a rectangular plate on a curved surface.
c Methods Computational used in Modeling.
      project program
      real dx,x,mi,E,l,q,tc,s,mo,f,D,v,h,vp
      real w(500),m(500)
      integer n,i
      i=1
      x=0
      w(i-1)=0
      m(i-1)=0
       write(*,*)'******************************************************'
       write(*,*)'THIS PROGRAM SOLVES THE DEFORMATION EQUATION '
       write(*,*)'OF A RECTANGULAR PLATE BY FINITE DIFFERENCES '
      write(*,*)''
       write(*,*)'******************************************************'
      write(*,*)''
650 write(*,*)'Choose a job option '
      write(*,*)''
      write(*,*)'Error 1 '
      write(*,*)'Board 2 '
      read(*,*)vp
      if (vp==1) then
       goto 660
      else
       if (vp==2) then
        goto 670
       else
        goto 650
670    end if
660   end if

600 write(*,*)'Choose a job option '
      write(*,*)''
      write(*,*)'Uniform charge. distributed 1 '
      write(*,*)'Uniform charge. distributed + axial load 2 '
      write(*,*)'Uniform charge. distributed + axial load + Moment 3 '
      read(*,*)tc
      if (tc==1)then
       goto 500
```

```
      else
       if (tc==2) then
        write(*,*)'Enter the axial load in KN '
        read(*,*)s
        goto 700
       else
        if (tc==3) then
         write(*,*)'Enter the axial load in KN '
         read(*,*)s
         write(*,*)'Enter the moment at the supports in KN-m '
         read(*,*)mo
         goto 800
        else
         goto 600
800     endif
700    endif
500   endif
      write(*,*)'Enter the number of data you want to work with'
      write(*,*)'to create a Deformation Vs X graph'
      write(*,*)''
      read(*,*)n
      write(*,*)'Between space interval dx '
      read(*,*)dx
      write(*,*)'Enter the length of the plate L (m)'
      read(*,*)l
      write(*,*)'Type the uniform charge. distributed q (KN/m) '
      read(*,*)q
      write(*,*)'Enter the Modulus of Elasticity E (KN/m2) '
      read(*,*)E
      write(*,*)'Enter the Moment of Inertia I (m4) '
      read(*,*)mi
      write(*,*)'Type the Poisson Ratio v '
      read(*,*)v
      write(*,*)'Enter the thickness of the plate h (m) '
      read(*,*)h

c create file for x-axis values with dx increments
      do i=1,n-1
       if(x==0)then
        goto 100
       else
100 open(60,file='ejex.dat',status='new')
        if(i==1)then
         x=x+dx
         goto 200
        else
200       m(i)=q*(l*x-x**2)/2
         write(60,*)x,m(i)
         x=x+dx
        endif
       end if
      end do
      close(60)
      D=E*h**3/(12*(1-v**2))
c an initial value is given to the variables w(i) to start the
c iterative process of convergence of the equations of the system
      do i=1,n
       if (i==n) then
```

```
        w(i)=0
        goto 300
      else
        w(i)=-0.0001
300     end if
      end do

c starts the numerical solution of the model and creates the file
c for said numerical solution

      DO K=1,100*n
        do i=1,n-1
          IF (i==1)then
            w(i-1)=0
            goto 1000
          ELSE
            if (i==n-1) then
              w(n)=0
             goto 900
            else
900           end if
1000           if (tc==1) then
             if (vp==1) then
               w(i)=0.5*((dx**2)*m(i)/(E*mi)+w(i-1)+w(i+1))
              goto 1010
             else
                     w(i)=0.5*((dx**2)*m(i)/D+w(i-1)+w(i+1))
1010            end if
            goto 1100
             else
             if (tc==2) then
              if (vp==1) then
               f=1/(2+s*(dx**2)/(E*mi))
                    w(i)=f*((dx**2)*m(i)/(E*mi)+w(i-1)+w(i+1))
               goto 1020
              else
               f=1/(2+s*(dx**2)/D)
                     w(i)=f*((dx**2)*m(i)/D+w(i-1)+w(i+1))
1020           end if
              goto 1200
             else
             if (tc==3) then
              if (vp==1) then
                f=1/(2+s*(dx**2)/(E*mi))
                  w(i)=f*((dx**2)*(m(i)+mo)/(E*mi)+w(i-1)+w(i+1))
                goto 1030
              else
                f=1/(2+s*(dx**2)/D)
                   w(i)=f*((dx**2)*(m(i)+mo)/D+w(i-1)+w(i+1))
1030           end if
              goto 1300
             else
1300          end if
1200          endif
1100          end if
           END IF
          end do
        END DO
```

```
    open(10,file='deformacion.xls',status='new')
    do i=1,n
      if (i==1) then
        w(i-1)=0
        write(10,30)w(i-1)
        goto 2000
      else
2000w(i)=-1*w(i)
        write(10,30)w(i)
30        format(D10.4)
      end if
    end do
    end
```

5. ANALYTICAL METHOD: DIFFERENTIAL EQUATION FOR DEFLECTION OF SOLID SLAB

The theory ofbending cylindrical slab, begins with the bending of a rectangular plate that is subjected to a load uniformly distributed along its length. The deflection of the plate surface can be assumed to be cylindrical. The solution of the problem can be centered on the doubling of an elementary strip of the plate, with a unit thickness and that is in the direction of the length of the plate. The deflection of that strip is given by the differential equation, which is similar to the equation for the deflection of a beam.

To obtain the deflection equation, consider the plate with uniform thickness, equal to h, and take the plate in the xy plane. For the solution of the plate only with the uniformly distributed load, the double integral method is used to determine the deflection equation along the plate, which is similar to that of a beam. The only thing that changes is the D value, which is determined by means of the equation seen in the numerical method, and it is (Thimoshenko Y Young, 1973):

$$D = \frac{Eh^3}{12(1-\upsilon^2)} \qquad (16)$$

Now, when the plate is subjected to a uniformly distributed load plus axial tension load, the moment at a distance x from the end of the section changes to:

$$M = \frac{ql}{2}x - \frac{qx^2}{2} - Sw \qquad (17)$$

Substituting this expression in equation 16, it remains:

$$\frac{dw^2}{dx^2} - \frac{Sw}{D} = -\frac{qlx}{2D} + \frac{qx^3}{2D} \qquad (a)$$

Introducing the notation:

$$\frac{Sl^2}{D4} = \mu^2 \qquad (18)$$

The general solution given by equation (a) can be written in the form:

$$w = C_1 senh\frac{2ux}{l} + C_2\cosh\frac{2ux}{l} + \frac{ql^3x}{8\mu^2 D} - \frac{ql^2x^2}{8u^2 D} - \frac{ql}{16u^2 D}$$

(b)

The integration constants C1 and C2 are determined by the boundary conditions of the structure, where the expression for the deflection of the structure is:

$$w = \frac{ql^4}{16\mu^4 D}\left(\frac{1-\cosh 2\mu}{senh 2\mu} senh\frac{2\mu x}{l} + \cosh\frac{2\mu x}{l} - 1\right) + \frac{ql^3x}{8u^2 D} - \frac{ql^2x^2}{8\mu^2 D}$$

(19)

And substituting hyperbolic cosine expressions, we get

$$w = \frac{ql^4}{16\mu^4 D}\left[\frac{Cosh(\mu(1-\frac{2x}{l}))}{\cosh\mu} - 1\right] + \frac{ql^2x}{8\mu^2 D}(l-x)$$

(20)

From equation 20, the deflection of a solid plate with uniform thickness can be determined.

6. ELEMENTOS FINITOS: STAAD-PRO

For the numerical solution by finite elements, the structure analysis and design program STAAD-PRO was used, due to its efficiency in this type of exercise. Next, in figures 4, 5 and 6, the input conditions and the results produced by the program are shown.

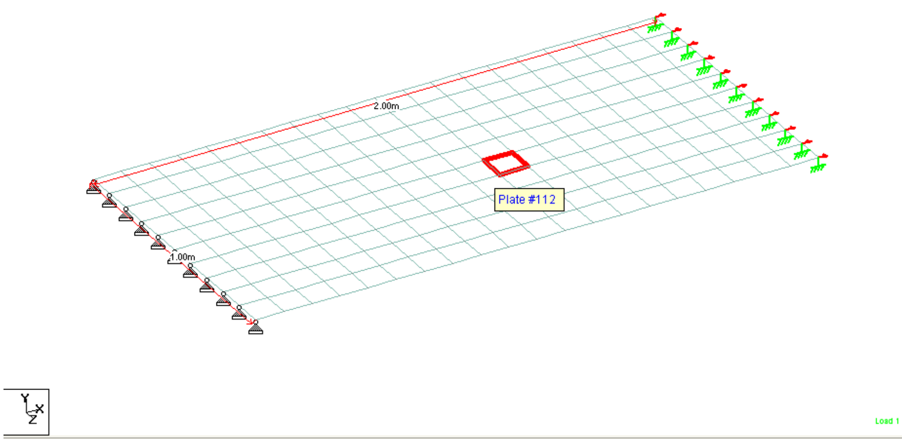

Figure 4. Mesh of the slab under study.

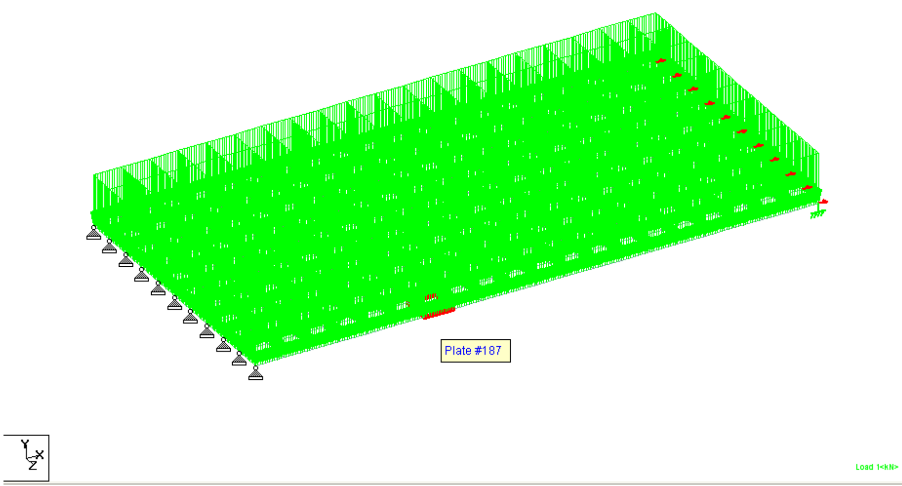

Figure 5. Loads on the plate.

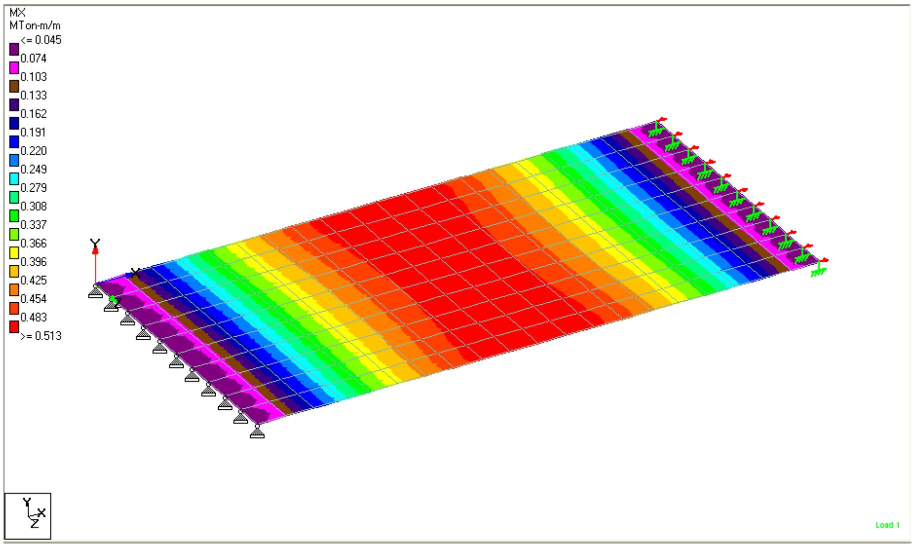

Figure 6. Resulting bending moment.

7. COMPARISON OF THE THREE METHODS

The methodology used to solve the problem for each one of the methods, consisted of analyzing a solid plate in one direction of 2.00 m wide by 2.00 m long, by 10 cm thick, whose characteristics were previously shown, in the approach of the problem; the plate is discretized every 0.10 m, then the problem was analyzed by the analytical method, using the equations shown, and finally, it was analyzed by the Staad-Pro structural program. The results by the three methods are shown below, in Table 1 and Figure 7.

Table 1. Deflection of a solid plate with a uniformly distributed load of 10 KN/m

DISTANCE (m)	**DEFLECTION (in meters)**		
	ANALYTIC	**Finite differences**	**Finite elements**
0	0	0	0
0,1	-0,00019106	-0,00019152	-0,000195
0,2	-0,0003767	-0,00037757	-0,000384
0,3	-0,00055202	-0,00055325	-0,000563
0,4	-0,0007127	-0,00071424	-0,000727
0,5	-0,000855	-0,0008568	-0,000873
0,6	-0,00097574	-0,00097776	-0,000996
0,7	-0,00107234	-0,00107453	-0,001095
0,8	-0,00114278	-0,00114509	-0,001166
0,9	-0,00118562	-0,001188	-0,00121

DISTANCE (m)	DEFLECTION (in meters)		
	ANALYTIC	Finite differences	Finite elements
1	-0,0012	-0,0012024	-0,001225
1,1	-0,00118562	-0,001188	-0,00121
1,2	-0,00114278	-0,00114509	-0,001167
1,3	-0,00107234	-0,00107453	-0,001095
1,4	-0,00097574	-0,00097776	-0,000996
1,5	-0,000855	-0,0008568	-0,000873
1,6	-0,0007127	-0,00071424	-0,000727
1,7	-0,00055202	-0,00055325	-0,000563
1,8	-0,0003767	-0,00037757	-0,000384
1,9	-0,00019106	-0,00019152	-0,000195
2	0	0	0

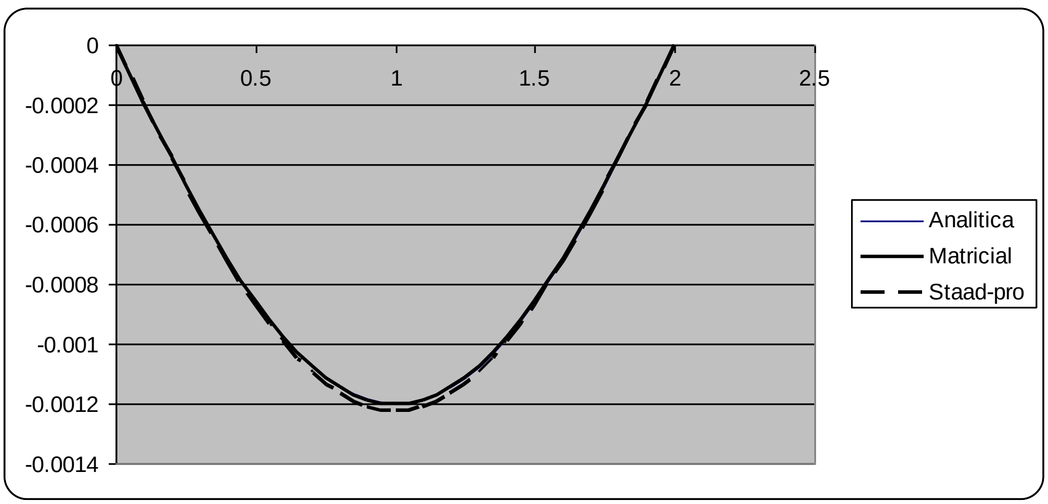

Figure 7. Deflection of a solid plate with a distributed load of 10 KN/m

Then, the same plate was analyzed by the three methods, but now with a uniformly distributed load of 10 KN/m, plus a compressive axial load of 10 KN. The discretization is the same of 0.10 m. The results obtained are shown in Table 2 and Figure 8.

Table 2. Deflection of a solid plate with a uniformly distributed load of 10 KN/m and a compressive axial load of 10 KN

DISTANCE (m)	DEFLECTION (in meters)		
	ANALYTIC	Finite differences	Finite elements
0	0	0	0
0,1	-0,00019062	-0,00019167	-0,000195
0,2	-0,00037584	-0,00037787	-0,000384
0,3	-0,00055075	-0,00055369	-0,000563
0,4	-0,00071105	-0,00071482	-0,000727
0,5	-0,00085302	-0,00085749	-0,000873
0,6	-0,00097347	-0,00097855	-0,000996
0,7	-0,00106984	-0,0010754	-0,001095
0,8	-0,00114012	-0,00114602	-0,001166
0,9	-0,00118285	-0,00118897	-0,00121
1	-0,0011972	-0,00120338	-0,001225
1,1	-0,00118285	-0,00118897	-0,00121
1,2	-0,00114012	-0,00114602	-0,001167
1,3	-0,00106984	-0,0010754	-0,001095
1,4	-0,00097347	-0,00097855	-0,000996
1,5	-0,00085302	-0,00085749	-0,000873
1,6	-0,00071105	-0,00071482	-0,000727
1,7	-0,00055075	-0,00055369	-0,000563
1,8	-0,00037584	-0,00037787	-0,000384
1,9	-0,00019062	-0,00019167	-0,000195
2	4,4409E-16	0	0

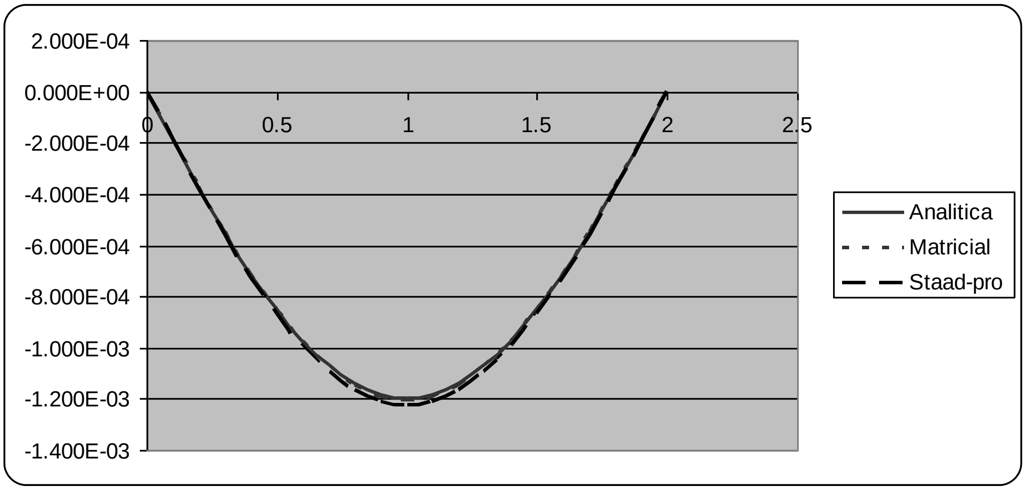

Figure 8. Deflection of solid plate with distributed load of 10 KN/m and axial load of 10 KN

8. CONCLUSIONS

The analysis of the problem developed for each one of the aforementioned methods, determines that the finite difference method, being much more elaborate in its development, is the one that comes closest to the numerical solution; however, the approximate finite element method, using Staad-pro, gives a good approximation. This demonstrates the effectiveness of any of the methods, as long as the most appropriate discretization is used, in which the degree of uncertainty is acceptable.

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