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Linear Fractional Maps That Induce Compact Linear Operators

Zakieldeen Aboabuda Mohammed Alhassn Ali, Sara Mustafa Alshareef Gesmallh, Amna Mahmoud Ahmed Bakhit, Manal Yagoub Ahmed Juma, Isra Abdalhleem Hassan Ali, Arafa .A.Alrhim Ahmed Dawood

Deanship of the Preparatory Year, Prince Sattam Bin Abdulaziz University, Alkharj, Saudi Arabia.
Email: Z.alhassan@psau.edu.sa
Correspondents Authors: Zakieldeen Aboabuda Mohammed Alhassn Ali
Z.alhassan@psau.edu.sa

ABSTRACT

It is generally known that the difference of two composition operators formed by linear fractional self-maps of a ball cannot be nontrivially compactly contained in the Hardy space or any common weighted Bergman space. This study extends this finding in two important ways: Inducing maps are expanded to linear fractional maps that carry a ball into a second, and the difference is extended to generic linear combinations, potentially higher-dimensional space.

Keywords Fractional maps, Composition Operators, Compact Linear Operators.

1. INTRODUCTION

When m is a non-negative integer, we designate $\mathbb{B}_m$ as the complex m -unit $\mathbb{C}^m$space's ball and $\mathbb{S}_m$ as the unit sphere that forms $\mathbb{B}_m$ border. To emphasize the special function played by the case where $m = 1$, we shall substitute the notations $\mathbb{D}$ and $\mathbb{T}$ for $\mathbb{B}_1$ and $\mathbb{S}_1$, respectively. This paper's main focus is on composition operators generated by linear fractional maps that move a ball into a possible different-dimensional space. We reserve a pair of arbitrary two dimensions, m and n, unless otherwise specified. The first function spaces that come to mind are the Hardy space and the weighted Bergman spaces.

Let the normalized $1+\varepsilon$ weighted volume measure $\mathbb{B}_m$be denoted by the notation $\varepsilon > -2$, $dv_{m,1+\varepsilon}$

$$dv_{m,1+\varepsilon}(z) := c_{m,1+\varepsilon}(1-|z|^2)^{1+\varepsilon}dv_m(z)$$

where $c_{m,1+\varepsilon}$ is the chosen constant to ensure that $v_{m,1+\varepsilon}(\mathbb{B}_m) = 1$ and dv_m is the normalized volume measure on $\mathbb{B}_m$. The holomorphic functions f on $\mathbb{B}_m$ Hilbert space ,is then the weighted Bergman space $A^2_{1+\varepsilon}(\mathbb{B}_m)$.

$$\|f\|_{m,1+\varepsilon} := \left\{\int_{\mathbb{B}_m} |f(z)|^2\, dv_{m,1+\varepsilon}(z)\right\}^{1/2}$$

has a limit. For the Hardy space $H^2(\mathbb{B}_m)$, This is the Hilbert space of all holomorphic functions f on B m when the norm equals 1, we use the idea $A^2_{-1}(\mathbb{B}_m)$) when

$\varepsilon = -2.$

$$\|f\|_{m,-1} := \left\{\sup_{0<r<1}\int_{\mathbb{S}_m} |f(r\zeta)|^2\, d\sigma_m(\zeta)\right\}^{1/2}$$

has a limitation. The normalized surface area measurement on $\mathbb{S}_m$ is indicated here by the symbol $d\sigma_m$.The weak star convergence $dv_{m,1+\varepsilon} \to d\sigma_m$ as $\varepsilon \to -2^+$ justifies the notation $A^2_{-1}(\mathbb{B}_m) = H^2(\mathbb{B}_m)$. A composition operator C_Φ induced by the holomorphic map $\Phi : \mathbb{B}_n \to \mathbb{B}_m$ is defined by

$$C_\Phi f := f \circ \Phi$$

for holomorphic f on $\mathbb{B}_m$ functions. To translate holomorphic functions on $\mathbb{B}_m$ to those on $\mathbb{B}_n$, C_Φ is a linear operator. Over the past few decades, various elements of these composition operators

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have been explored; for a summary of the work done before the mid-1990s, also the monographs by Shapiro [17] and Cowen-MacCluer [3]. Comparing the several-variable theory of composition operators to the one-variable theory reveals how much more complicated it is, as is common knowledge.

For example, Littlewood's Subordination Principle has the well-known consequence that, when $m = n = 1$, C_Φ is always confined on the Hardy space and the weighted Bergman spaces. Such boundedness is no longer guaranteed when higher dimensional balls are used; for further information, see [3, Section 6.3] for the Hardy space and [9] for the weighted Bergman spaces. The existence of bounded composition operators results from holomorphic self-maps of a ball satisfying a particular additional property, though.

The so-called Wogen condition is one such additional attribute; for further information, see [3, Section 6.2] and [19]. Cowen and MacCluer discovered linear fractional maps, another kind of inducing functions that guarantee boundedness, in a quite different setting. In this context, we refer to a linear fractional map as one that has the form $\Phi : \mathbb{B}_m \to \mathbb{C}^m$, and Φ is holomorphic.

$$\Phi(z) = \frac{Az + b}{\langle z, c\rangle + d} \qquad (1)$$

When the linear operator $A : \mathbb{C}^n \to \mathbb{C}^m$, $b \in \mathbb{C}^m$, $c \in \mathbb{C}^n$ and $d \in \mathbb{C}$ are present. Here, $\langle \cdot,\cdot \rangle$ stands for the common inner product on $\mathbb{C}^n$, For example, $\langle z, w\rangle$ is equal to $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + \overline{z}_n\overline{w}_n$ for $z, w \in \mathbb{C}^n$ where z_j stands for the j-th component of z; the context should make it apparent which dimension is lacking in this notation.

There is no denying that on the set of $z \in \mathbb{C}^n$ with $\langle z, c\rangle + d \neq 0$ extends to a holomorphic function. When $c = 0$, Take note that at $z = -dc/|c|^2$, the denominator of Φ disappears. As a result, when $\Phi(\mathbb{B}_n) \subset \mathbb{B}_m$ is added, either $||d| > |c|$ or Φ is obtained or reduces to a constant map. Therefore, we can assume $|d| > |c|$in (18) from the beginning when $\Phi(\mathbb{B}_n) \subset \mathbb{B}_m$. In particular, we observe that in an open set containing $\mathbb{B}_n$, any linear fractional map $\Phi : \mathbb{B}_n \to \mathbb{B}_m$ is holomorphic. When $m = n$, Cowen and MacCluer [4] initially introduced and researched linear fractional maps and associated composition operators. They showed that composition operators generated by linear fractional self-maps of a ball are always bounded in terms of boundedness on the Hardy space and the weighted Bergman spaces (see [4, Theorems 14 and 15]). This turns out to be true for universal linear fractional maps from one ball into another, provided the weight parameters are coupled properly. In order to be more specific, we demonstrate (see Theorem (4.2.7)) that whenever $m = n + \mathcal{E}$, any linear fractional map $\Phi : \mathbb{B}_n \to \mathbb{B}_m$ generates a bounded composition operator $C_\Phi : A^2_{1+\varepsilon}(\mathbb{B}_m) \to A^2_{1+2\varepsilon}(\mathbb{B}_n)$This justifies the parameter connection that is imposed on the premises of our main result, Theorem(1.1), below.

In terms of compactness, we note that $C_\Phi : A^2_{1+\varepsilon}(\mathbb{B}_m) \to A^2_{1+2\varepsilon}(\mathbb{B}_n)$ with $m = n + \mathcal{E}$ is compact if and only if $\|\Phi\|_\infty := \sup_{\zeta\in\mathbb{S}_n}|\Phi(\zeta)| < 1$. The requirement is not difficult.

In to show the need, we note that if it is closed and bounded, a simple modification of the argument of [3, Theorem 3.43] results in $(\frac{1-|\Phi(r\zeta)|^2}{1-r^2} \to \infty$ as $r \to 1^-$for each $\zeta \in \mathbb{S}_n$ and subsequently $\|\Phi\|_\infty < 1$ by smoothness of on Φ on $\overline{\mathbb{B}}_m$.

Research on compact differences, or more generally, linear combinations, in the theory of composition operators, has recently attracted attention. For examples, see [5, 7, 14, 18] for the Hardy spaces and [1, 2, 6, 8, 10, 11, 13, 15, 16] for the weighted Bergman spaces.

Composition operators generated by linear fractional self-maps of a ball are not able to construct a nontrivial compact difference, as independently shown in [6] and [8]. These operators are known to behave quite tightly in this circumstance.

We broaden this rigidity in two important dimensions. In particular, we broaden the definition of difference to include linear combination and, concurrently, to include linear fractional self-maps of a ball that take a ball into another. The following theorem provides a clearer explanation of our main discovery.

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Theorem(1.1) Given a non-negative integer N, let $\Phi^1, \dots, \Phi^N : \mathbb{B}_n \to \mathbb{B}_m$ distinctly different linear fractional maps and $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. let $\mathcal{E} \geq -1$, $m = n + \mathcal{E}$, assume that $\sum_{j=1}^{N} \lambda_j C_{\Phi^j} : A^2_{1+\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+2\mathcal{E}}(\mathbb{B}_n)$ is closed and bounded. Then, for each

$j = 1, \dots, N$, either $\|\Phi^j\|_\infty < 1$ or $\lambda_j = 0$.

We demonstrate several fundamental characteristics of linear fractional maps that are required for the theorem's proof (1.1). We demonstrate the theory (1.1). Our strategy differs significantly from that in [6] and [8]. We also note that the parameter relation $m = n + \mathcal{E}$ guarantees the boundedness of the composition operators under discussion (Theorem (3.1)).

2. FRACTIONAL LINEAR MAPS

We first examine the impact of linear fractional maps on horocycle radii. Then, we give a uniqueness result for maps of linear fractions that contain a ball colliding with another.

let $0 < t < \infty$, we denote by Δ_t the horodisk consisting of all points $\lambda \in \mathbb{D}$ satisfying

$$|1-\lambda|^2 < t(1-|\lambda|^2).$$

A simple computation can be used to verify that $\Delta_t \subset \mathbb{D}$ is truly a disk with a radius of $\frac{t}{t+1}$ and a center at $\frac{1}{t+1}$. To be more specific, Δ_t is tangent to $\mathbb{T}$ at 1.

As t rises to 1, Δ_t also grows and fills the entire $\mathbb{D}$. The horocycle that forms the boundary of t is denoted by the symbol Δ_t. Since Γ_t is perpendicular to $\mathbb{T}$ at 1, one may verify that

$$\lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{(1-\lambda)^2}{|1-\lambda|^2} = -1. \qquad (2)$$

We just enter$\Delta_\infty := \mathbb{D}$ and $\Gamma_\infty := \mathbb{T}$ for $t = \infty$.

Note the Hopf Lemma in the following proposition: $\varphi'(1) > 0$.

Proposition (2.1):

If $\varphi(1) = 1$, then if $0 < t \leq \infty$, $\varphi : \Delta_t \to \mathbb{D}$ fractional linear map. Then s is specified by the equation and $\varphi(\Gamma_t) = \Gamma_s$.

$$1 + \frac{1}{s} = \frac{1}{\varphi'(1)}\left\{1 + \frac{1}{t} + \frac{Re[\varphi''(1)]}{\varphi'(1)}\right\} \qquad (3).$$

Proof:

Since $\varphi : \Delta_t \to \mathbb{D}$ is a linear fractional map $\varphi(1) = 1$, the horocycle $\Gamma_t$is transferred onto another horocycle. Thus, we just need to determine the horocycle's radius, or $\varphi(\Gamma_t)$. To accomplish this, We parameterize the curve for $\varphi(\Gamma_t)$.

$\tau(\theta) := \left(1 - r + re^{i\theta}\right)$, $-\pi \leq \theta \leq \pi$ where $\frac{1}{r} := 1 + \frac{1}{t}$.

Because $\varphi(\Gamma_t)$ is a circle passing 1 at $\theta = 0$, it is sufficient to show that the right-hand side of (3) equals the curvature of τ at $\theta = 0$.

Since

$$\tau'(0) = r\varphi'(1)i \quad and \quad \tau''(0) = r\varphi'(1) - r^2\varphi''(1).$$

Since the curve's normal vector is 1 is $(-1,0)$, we can see that the acceleration vector $\tau''(0)$ is normal component is $r\varphi'(1) + r^2 Re[\varphi''(1)]$; keep in mind that $\varphi'(1) > 0$.

We have specifically'$\varphi'(1) + rRe[\varphi''(1)] \geq 0$. The curvature of at τ at $\theta = 0$is therefore given by

$$\frac{|Im[\tau'(0)] \times Re[\tau''(0)]|}{|\tau'(0)|^3} = \frac{\varphi'(1) + rRe[\varphi''(1)]}{r[\varphi'(1)]^2}$$

$$= \frac{1}{\varphi'(1)}\left\{\frac{1}{r} + \frac{Re[\varphi''(1)]}{\varphi'(1)}\right\}.$$

The evidence is complete since $\frac{1}{r} = 1 + \frac{1}{t}$.

While not necessary for the current paper, we note several Proposition (2.1) ramifications that may be of interest on their own.

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Remark (2.2):

(1) Given that $\varphi(1)=1$, let $\mathbb{D}$ be a linear fractional self-map. For t = 0, be. By Proposition for $0<t<1$, we have (2.1)

$$\frac{1-|\varphi(z)|^2}{1-|z|^2} = \frac{1-|\varphi(z)|^2}{|1-\varphi(z)|^2}\cdot\left|\frac{1-\varphi(z)}{1-z}\right| \cdot\frac{|1-z|^2}{1-|z|^2}$$

$$=\frac{t}{\varphi'(1)^2}\left\{\left(1+\frac{1}{t}\right)\varphi'(1)-\varphi'(1)^2 + Re[\varphi''(1)]\right\}\left|\frac{1-\varphi(z)}{1-z}\right|^2$$

for $z\in\mathbb{D}\cap\Gamma_t$. This yields

$$\lim_{\substack{z\to1\\ z\in\Gamma_t}}\frac{1-|\varphi(z)|^2}{1-|z|^2} = \varphi'(1) + t\{\varphi'(1)-\varphi'(1)^2 + Re[\varphi''(1)]\}. \qquad (4)$$

As we wait, we have

$$\lim_{\substack{z\to1\\ z\in\Gamma_t}}\frac{1-|\varphi(z)|^2}{1-|z|^2}\ge \liminf_{z\to1}\ \frac{1-|\varphi(z)|^2}{1-|z|^2} = \varphi'(1);$$

The Julia-Carathéodory Theorem guarantees the final equality. This results in the inequality along with step (4).

$$Re[\varphi''(1)] \ge \varphi'(1)^2-\varphi'(1). \qquad (5)$$

From (4) and Proposition (2.1), we can see that the equality condition in (5) must be true.

$$\lim_{\substack{z\to1\\ z\in\Gamma_t}}\frac{1-|\varphi(z)|^2}{1-|z|^2} = \varphi'(1)\ \ \frac{forsome}{all}0<t<\infty$$

$$\Leftrightarrow Re[\varphi''(1)] = \varphi'(1)^2-\varphi'(1)$$

$$\Leftrightarrow \varphi(\Gamma_\infty)=\Gamma_\infty$$

$$\Leftrightarrow \varphi\ is\ an\ automorphism\ of\ \mathbb{D}.$$

(2) We also notice that the inequality (5) holds for general holomorphic self-maps if and only if derivatives up to the second order are comprehensible. Take into account any holomorphic self-map φ of $\mathbb{D}$ that admits a form expansion and is twice continuously differentiable near 1.

$$\varphi(z)=1+\varphi'(1)(z-1)+\frac{\varphi''(1)}{2}(z-1)^2 + o(|z-1|^2)$$

as $z\to 1$within $\mathbb{D}$. Note that the image curve $\varphi(\Gamma_t)$ is tangent to $\mathbb{T}$ for every $0<t<1$and that its curvature at 1 is at least 1. In light of Proposition (2.1)'s proof, we so arrive at

$$\frac{1}{\varphi'(1)}\left\{1+\frac{1}{t}+\frac{Re[\varphi''(1)]}{\varphi'(1)}\right\}\ge 1$$

for every $0<t<1$. So, if we take the limit $t\to\infty$, we can see that (5) still holds for this generic. We will now discuss the uniqueness property for fractional linear maps that involve one ball being rolled into another. Some indication is necessary. the remaining portion of the paper, we

$$\boldsymbol{e}_n := (1,0,\dots,0)\in\mathbb{S}_n$$

as an accepted benchmark. The differentiation with regard to the j-th component of the supplied variable is denoted by ∂_j,and we put

$$\partial := (\partial_1,\dots,\partial_n)\ \ and\ \ \partial_{jk} := \partial_j\partial_k.$$

In addition, we write for a holomorphic map $\Phi:\mathbb{B}_n\to\mathbb{B}_m$

$$\Phi' := \left(\partial_j\Phi_k\right)_{m\times n}$$

for's complicated derivative Φ. Throughout, Φ_k stands for the k-th component function of Φ.

It is simple to observe how the second-order data at a particular place totally dictate a one-variable fractional linear map. We require such uniqueness in a multi-variable form.

Lemma(2.3): Let $\varphi:\mathbb{B}_n\to\mathbb{C}$ be fractional linear map that is holomorphic at $\boldsymbol{e}_n$. It follows that is constant if $\partial\varphi(\boldsymbol{e}_n)=0$.

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Proof: Since $\varphi : \mathbb{B}_n \to \mathbb{C}$ is fractional linear map, there are $a, c \in \mathbb{C}^n$ and $b, d \in \mathbb{C}$ such that

$$\varphi(z) = \frac{\langle z, a\rangle + b}{\langle z, c\rangle + d}. \quad (6)$$

We can make the assumption that $\overline{c}_1 + d \neq 0$; otherwise, because φ is holomorphic in the vicinity of $\boldsymbol{e}_n$, reduces to a constant map. A simple computation results in

$$\partial_j(z) = \frac{\overline{a}_j(\langle z, c\rangle + d)\overline{c}_j(\langle z, a\rangle + b)}{(\langle z, c\rangle + d)^2} \quad (7)$$

that is

$$\partial_j(\boldsymbol{e}_n) = \frac{\overline{a}_j(\overline{c}_1 + d) - \overline{c}_j(\overline{a}_1 + b)}{(\overline{c}_1 + d)^2} \quad (8)$$

for every j As a result, if $\partial\varphi(\boldsymbol{e}_n) = 0$, we get

$$\overline{a}_1 d = \overline{c}_1 b \quad and \quad \overline{a}_j = \frac{\overline{a}_1 + b}{\overline{c}_1 + d}\overline{c}_j = \varphi(\boldsymbol{e}_n)\overline{c}_j$$

for every j Note that $a = \overline{\varphi(\boldsymbol{e}_n)}c$. As a result, if $c_1 \neq 0$, then $b = \varphi(\boldsymbol{e}_n)d$. If not, we have $a_1 = c_1 = 0$ (recall $d \neq 0$) and $b = \varphi(\boldsymbol{e}_n)d$ again. In either scenario, the answer is $\varphi = \varphi(e_n)$.The evidence is conclusive. The following lemma's characteristics (a) and (b) are true for general holomorphic mappings $\varphi : \mathbb{B}_n \to \mathbb{D}$ of class C^3 on $\overline{\mathbb{B}}_n$; see [3, Lemma 6.6]. The following proof of proposition (2.5) will make use of property (a). The next section's (25) and (26) employ the properties (b) and (c), which are empty for $n = 1$.

Lemma(2.4): Given that $\varphi(\boldsymbol{e}_n) = 1$, let $\varphi : \mathbb{B}_n \to \mathbb{D}$ be a linear fractional map. Then, the following claims are true:

(a) $\partial_1\varphi(\boldsymbol{e}_n) > 0$;
(b) $\partial_j\varphi(\boldsymbol{e}_n) = 0$ for $j = 2, \dots, n$;
(c) $\partial_{jk}\varphi(\boldsymbol{e}_n) = 0$ for $j, k = 2, \dots, n$.

Proof: Only proof remains (c). Allow φ to be as in (6). Since

$$1 = \varphi(\boldsymbol{e}_n) = \frac{\overline{a}_1 + b}{\overline{c}_1 + d}, \quad (9)$$

As of now (8)

$$\partial_j\varphi(\boldsymbol{e}_n) = \frac{\overline{a}_j - \overline{c}_j}{\overline{c}_1 + d} \quad (10)$$

for every j .As a result, by (b)

$$a_j = c_j, \ j = 2, \dots, n. \quad (11)$$

So, by (7), (9) and for $j = 2, \dots, n$,and (11)

$$\partial_j\varphi(z) = \overline{c}_j \frac{\langle z, c - a\rangle + d - b}{(\langle z, c\rangle + d)^2} = \overline{c}_j(\overline{c}_1 - \overline{a}_1)\frac{z_1 - 1}{(\langle z, c\rangle + d)^2}. \quad (12)$$

Applying ∂_k to both sides of the above equation for $k = 2, \dots, n$ and then evaluating at $z = \boldsymbol{e}_n$ leads us to our conclusion (c). The evidence is conclusive. Here is an example of the following uniqueness property for linear fractional maps.

Proposition(2.5): If $\Phi, \Psi : \mathbb{B}_n \to \mathbb{B}_m$ are fractional linear maps.

$\Phi(\boldsymbol{e}_n) = \Psi(\boldsymbol{e}_n) = \boldsymbol{e}_m$. If

$$\Phi'(\boldsymbol{e}_n) = \Psi'(\boldsymbol{e}_n) \quad and \quad \partial(\partial_1\Phi_1)(\boldsymbol{e}_n) = \partial(\partial_1\Psi_1)(\boldsymbol{e}_n),$$

Then

$\Phi = \Psi$.

Proof: First, we prove $\Phi_1 = \Psi_1$. Let $\varphi := \Phi_1$ and $\psi := \Psi_1$ for short. Choose $a, a', c, c' \in \mathbb{C}^m$ and $b, b', d, d' \in \mathbb{C}$ such that

$$\varphi(z) = \frac{\langle z, a\rangle + b}{\langle z, c\rangle + d} \quad and \quad \psi(z) = \frac{\langle z, a'\rangle + b'}{\langle z, c'\rangle + d'}.$$

Let φ_{e_n} and ψ_{e_n} be the slice functions provided by

$$\varphi_{e_n}(\lambda) := \varphi(\lambda_{e_n}) = \frac{\overline{a}_1\lambda + b}{\overline{c}_1\lambda + d} \quad and \quad \psi_{e_n}(\lambda) := \psi(\lambda_{e_n}) = \frac{\overline{a}'_1\lambda + b'}{\overline{c}'_1\lambda + d'}$$

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for $\lambda \in \mathbb{D}$. As a result, φ_{e_n} and ψ_{e_n}) are linear fractional self-maps of $\mathbb{D}$, and

$$\varphi_{e_n}(1) = \varphi(\boldsymbol{e}_n) = \psi(\boldsymbol{e}_n) = \psi_{e_n}(1),$$

$$\varphi'_{e_n}(1) = \partial_1\varphi(\boldsymbol{e}_n) = \partial_1\psi(\boldsymbol{e}_n) = \psi'_{e_n}(1),$$

$$\varphi''_{e_n}(1) = \partial_{11}\varphi(\boldsymbol{e}_n) = \partial_{11}\psi(\boldsymbol{e}_n) = \psi''_{e_n}(1).$$

Because a linear fractional self-map of $\mathbb{D}$ at a particular point is completely governed by its second-order data, this means that $\varphi_{e_n} = \psi_{e_n}$. Consequently, we may assume scaling coefficients as necessary.

$$a_1 = a_1', \ b = b', \ c_1 = c_1' \ \ and \ \ d = d'. \quad (13)$$

Assuming that $\varphi(\boldsymbol{e}_n) = \psi(\boldsymbol{e}_n) = 1$, we also obtain by (11)

$$a_j = c_j \quad and \quad a_j' = c_j', \ j = 2, \dots, n.$$

Now, it is sufficient to demonstrate that $\varphi = \psi$.

$$c_j = c_j', \ j = 2, \dots, n \quad (14)$$

Fix an arbitrary $j = 2, \dots, n$ for the remainder of the proof to demonstrate this. By using ∂_1 to apply to (12) and evaluating at z = $\boldsymbol{e}_n$, we get

For the remainder of the proof, fix an arbitrary $j = 2, \dots, n$. By adding ∂_1 to (12) and evaluating at z = $\boldsymbol{e}_n$, we obtain.

$$\partial_{j1}\varphi(\boldsymbol{e}_n) = \partial_{1j}\varphi(\boldsymbol{e}_n) = \frac{(\overline{c}_1 - \overline{a}_1)}{(\overline{c}_1 + d)^2}\overline{c}_j.$$

In a similar vein, we

$$\partial_{j1}(\boldsymbol{e}_n) = \frac{(\overline{c}_1' - \overline{a}_1')}{(\overline{c}_1' + d'')}\overline{c}_j' = \frac{(\overline{c}_1 - \overline{a}_1)}{(\overline{c}_1 + d)^2}\overline{c}_j',$$

The final equality originates from (13). According to the presumption

$\partial_{j1}\varphi(\boldsymbol{e}_n) = \partial_{j1}\psi(\boldsymbol{e}_n)$ that $(c_1 - a_1)(c_j - c_j') = 0$. As a result, we arrive at (14) as needed $a_1 \neq c_1$ by Lemma (2.4), (a), and (10). We can now see that Φ and Ψ and share a same denominator since $\Phi_1 = \Psi_1$. So, Additionally, $\Phi - \Psi : \mathbb{B}_n \to \mathbb{C}^m$ is a linear fractional map that is holomorphic in the vicinity of $\boldsymbol{e}_n$.

We arrive at the conclusion $\Psi = \Phi$ by lemma given that $\Phi'(\boldsymbol{e}_n) = \Psi'(\boldsymbol{e}_n)$ and $\Phi(\boldsymbol{e}_n) = \Psi(\boldsymbol{e}_n)$, respectively (2.3). The evidence is overwhelming.

3. OPERATORS OF COMPOSITIONS

Theorem (1.1) assumes that each of the composition operators under examination is individually bounded, thus to start, we demonstrate that the parameter connection in this statement is a natural one.

Theorem(3.1): If $\varphi : \mathbb{B}_n \to \mathbb{B}_m$be a fractional linear map. Then

$$C_\Phi : A^2_{1+\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+2\mathcal{E}}(\mathbb{B}_n)$$

is constrained whenever $m = n + \mathcal{E}$and $\mathcal{E} \geq -1$.

Proof: As stated in the Introduction, [4, Theorems 14 and 15] show the $m = n$. So let's say $m \neq n$. Replace $\mathcal{E} \geq -1$ with $m = n + \mathcal{E}$.

We first look at the situation $m > n$. Let $P_{m,n} : \mathbb{B}_m \to \mathbb{B}_n$, $P_{m,n}(z_1, \dots, z_m) := (z_1, \dots, z_n)$be the projection map. The famous integral identities come to mind when combined with this projection map.

$$\int_{\mathbb{B}_n} h(z)\, dv_{n,1+2\mathcal{E}}(z) = \int_{\mathbb{B}_m} h \circ P_{m,n}(w)\, dv_{m,1+\mathcal{E}}(w) \quad for\ \mathcal{E} > -2$$

and

$$\int_{\mathbb{B}_n} h(z)\, dv_{n,1+2\mathcal{E}}(z) = \int_{\mathbb{S}_m} h \circ P_{m,n}(\zeta)\, d\sigma_m(\zeta) \quad for\ \mathcal{E} = -2$$

regarding the functions $h \in L^1(dv_{n,1+2\mathcal{E}})$.A simple application of Fubini's Theorem can be used to verify the case $\mathcal{E} > -2$, the situation $\mathcal{E} = -2$ can is, for example, found in [19, Lemma 1.9]. These core identities serve as evidence for us that

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$$\|C_\Phi P_{m,n} f\|_{m,1+\mathcal{E}} = \|f\circ\Phi\circ P_{m,n}\|_{m,1+\mathcal{E}} = \|f\circ\Phi\|_{n,1+2\mathcal{E}} = \|C_\Phi f\|_{n,1+2\mathcal{E}}$$

$f \in A^2_{1+\mathcal{E}}(\mathbb{B}_m)$ and $\mathcal{E} \geq -2$for any. Because $\Phi \circ P_{m,n}$ is a linear fractional self-map of $\mathbb{B}_m$, $C_{\Phi\circ P_{m,n}} : A^2_{1+\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+\mathcal{E}}(\mathbb{B}_m)$ is also bounded.

In light of the foregoing, we deduce that $C_\varphi : A^2_{1+\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+\mathcal{E}}(\mathbb{B}_m))$ is bounded as necessary. We will now look at case $m < n$. In this instance, we employ the well-known Carleson measure method; for more information on Carleson measures, see, for instance, [3, Section 2.2]. We offer information for the case $\mathcal{E} > -1$; using [168, Theorem 2.38], the case $\mathcal{E} = -1$can be handled identically. Note

$$\int_{\mathbb{B}_n} |f\circ\Phi|^2\, dv_{n,1+2\mathcal{E}}(z) = \int_{\mathbb{B}_m} |f|^2\, d\left(v_{n,1+2\mathcal{E}}\circ\Phi^{-1}\right) \quad f \in A^2_{1+\mathcal{E}}(\mathbb{B}_m)$$

where the pullback measure is supplied by $v_{n,1+2\mathcal{E}}\circ\Phi^{-1}$.

$\left(v_{n,1+2\mathcal{E}}\circ\Phi^{-1}\right)(E) := v_{n,1+2\mathcal{E}}[\Phi^{-1}(E)]$ for Borel sets $E \subset \mathbb{B}_m$. This shows (see [3, Theorem 2.38]) that $C_\Phi : A^2_{1+\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+\mathcal{E}}(\mathbb{B}_m)$ is bounded if and only if $v_{n,1+2\mathcal{E}}\circ\Phi^{-1}$ is a Carleson measure for $A^2_{1+\mathcal{E}}(\mathbb{B}_m)$, which means

$$\sup_{\eta\in\mathbb{S}_m}\left(v_{n,1+2\mathcal{E}}\circ\Phi^{-1}\right)[S^m_\delta(\eta)] = \mathcal{O}\left(\delta^{m+2+\mathcal{E}}\right),\ \delta > 0 \tag{15}$$

where

$$S^m_\delta(\eta) := \{z \in \mathbb{B}_m : |1 - \langle z,\eta\rangle < \delta\}.$$

We now show (15). For that purpose, we consider the embedding map

$E_{m,n} : \mathbb{B}_m \to \mathbb{B}_n$ given by $E_{m,n}(z) := (z,0,\dots,0)$. This time $\widetilde{\Phi} := E_{m,n}$ is a linear fractional self-map of $\mathbb{B}_n$ and thus $C_{\widetilde{\Phi}} : A^2_{1+2\mathcal{E}}(\mathbb{B}_n) \to A^2_{1+2\mathcal{E}}(\mathbb{B}_n)$ is bounded, or equivalently, the pullback measure $v_{n,1+2\mathcal{E}}\circ\widetilde{\Phi}^{-1}$ is a Carleson measure for $A^2_{1+2\mathcal{E}}(\mathbb{B}_n)$. More explicitly, we have

$$\sup_{\eta\in\mathbb{S}_n}\left(v_{n,1+2\mathcal{E}}\circ\widetilde{\Phi}^{-1}\right)[S^m_\delta(\zeta)] = \mathcal{O}\left(\delta^{n+2+2\mathcal{E}}\right),\ \delta > 0. \tag{16}$$

Meanwhile, note

$$\widetilde{\Phi}^{-1}[S^m_\delta(\tilde{\eta})] = \widetilde{\Phi}^{-1}[S^m_\delta(\eta)] \tag{17}$$

for $\eta \in \mathbb{S}_n$ and $\tilde{\eta} := (\eta,0,\dots,0) \in \mathbb{S}_n$. Now, since $m = n + \mathcal{E}$, we see that (16) and (17) imply (15). The proof is strong. Now, let's move on to the theorem's proof (1.1). We require multiple introductions. We start by remembering the reproducing kernels for the spaces in question. Let $\mathcal{E} \geq -2$. Every $w \in \mathbb{B}_m$corresponds to a different replicating kernel, as is widely known.

$K^{m,1+\mathcal{E}}_w \in A^2_{1+\mathcal{E}}(\mathbb{B}_m)$ such that

$$f(w) = \langle f, K^{m,1+\mathcal{E}}_w\rangle_{A^2_{1+\mathcal{E}}(\mathbb{B}_m)},\quad f \in A^2_{1+\mathcal{E}}(\mathbb{B}_m)$$

where the inner product on $\langle\cdot,\cdot\rangle_{A^2_{1+\mathcal{E}}(\mathbb{B}_m)}$ is indicated by, $A^2_{1+\mathcal{E}}(\mathbb{B}_m)$. $K^{m,1+\mathcal{E}}_w$ formula is well-known and is provided by

$$K^{m,1+\mathcal{E}}(z,w) := K^{m,1+\mathcal{E}}_w(z,z) = \frac{1}{(1-\langle z,w\rangle)^{m+2+\mathcal{E}}}; \tag{18}$$

see, for example, [19]. Note

$$\left\|K^{m,1+\mathcal{E}}_w\right\|^2_{m,1+\mathcal{E}} = K^{m,1+\mathcal{E}}(z,z) = \frac{1}{(1-|z|^2)^{m+2+\mathcal{E}}}. \tag{19}$$

The following positive of the replicating kernels is presumably well known. Here, we offer a rather simple proof of completeness.

Lemma(3.2): Let $\mathcal{E} \geq -2$, a non-Negative integer N, let $z^1,\dots,z^N$ be distinct points in $\mathbb{B}_m$. Then

$$\sum_{j,k=1}^{N} \lambda_j\bar{\lambda}_k K^{m,1+\mathcal{E}}(z^j,z^k) \geq 0$$

for any choice of $\lambda_1,\dots,\lambda_N \in \mathbb{C}$. Additionally, the equality only applies when

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$\lambda_1 = \cdots = \lambda_N = 0.$

Proof: The reproducing property suggests that

$$K^{m,1+\varepsilon}(z,w) = \langle K_w^{m,1+\varepsilon}, K_z^{m,1+\varepsilon}\rangle_{A^2_{1+\varepsilon}(\mathbb{B}_m)}$$

for all $z, w \in \mathbb{B}_m$. Thus, we have

$$\sum_{j,k=1}^{N} \lambda_j \bar{\lambda}_k K^{m,1+\varepsilon}(z^j, z^k)$$
$$= \sum_{j,k=1}^{N} \langle \lambda_j K_{z^k}^{m,1+\varepsilon}, \lambda_k K_{z^k}^{m,1+\varepsilon}\rangle_{A^2_{1+\varepsilon}(\mathbb{B}_m)}$$

$$= \langle \sum_{k=1}^{N} \bar{\lambda}_k K_{z^k}^{m,1+\varepsilon}, \sum_{j=1}^{N} \bar{\lambda}_j K_{z^j}^{m,1+\varepsilon}\rangle_{A^2_{1+\varepsilon}(\mathbb{B}_m)}$$

$$= \left\| \sum_{j=1}^{N} \bar{\lambda}_j K_{z^j}^{m,1+\varepsilon} \right\|^2_{m,1+\varepsilon}$$

for any selection of $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. The first portion of the lemma is proven by this. Since the points $z^1, \dots, z^N$ are all distinct, it should be noted that $\{K_{z^1}^{m,1+\varepsilon}, \dots, K_{z^N}^{m,1+\varepsilon}\}$ are linearly independent. Consequently, the lemma's second portion is also true.

We now prove the following lemma, which is essential to our later arguments, using Lemma (3.2). We employ the standard multi-index notation both in the proof that follows and elsewhere. This is,

$$|\gamma| := \gamma_1 + \cdots + \gamma_m,$$

$$\gamma! := \gamma_1! \dots \gamma_m!,$$

$$x^\gamma := x_1^{\gamma_1} \dots x_m^{\gamma_m}$$

For $x = (x_1, \dots, x_m)$ and m -tuples $\gamma = (\gamma_1, \dots, \gamma_m)$ Among integers, non-negative;

These notations should make it clear from the context which dimensions are involved. Naturally, it is expected that 1 represents 0^0.

Lemma(3.3): A von-negative integer is given$N$, let $(\mu_1, B_1), \dots, (\mu_N, B_N)$ be distinct points in $\mathbb{C} \times \mathbb{C}^m$. If $\lambda_1, \dots, \lambda_N$ complicated numbers in such a form

$$\sum_{j,k=1}^{N} \lambda_j \bar{\lambda}_k \left(\mu_j + \bar{\mu}_k + \langle B_j, B_k\rangle\right)^s = 0 \qquad (20)$$

for all integers $s \geq 0$, then $\lambda_1 = \cdots = \lambda_N = 0$.

be complex numbers that satisfy (20) for all $s \geq$ 0integers.

Proof: Let $\lambda_1, \dots, \lambda_N$ be complex numbers that satisfy (20) for all $s \geq 0$ integers. First, we say

$$\sum_{j=1}^{N} \lambda_j \mu_j^p B_j^\gamma = 0 \qquad (21)$$

for every single integer $p \geq 0$and several indices.

By the instance $s = 0$ of (21) it is clear that it holds for $p = 0$ and $|\gamma| = 0$. (20). The next step is to induct on $2p + |\gamma|$. Therefore, let's assume that (38) is true whenever $2p + |\gamma| \leq s - 1$ for an integer $s \geq 1$. Take note of (20) that

$$0 = \sum_{j,k=1}^{N} \lambda_j \bar{\lambda}_k \left(\mu_j + \bar{\mu}_k + (B_j, B_k)\right)^s$$

$$= \sum_{p+q+r=s} \frac{s!}{p!\,q!\,r!} \sum_{j,k=1}^{N} \lambda_j \bar{\lambda}_k \mu_j^p \bar{\mu}_k^q \langle B_j, B_k\rangle^r$$

$$= \sum_{p+q+r=s} \frac{s!}{p!\,q!} \sum_{|\gamma|=r} \frac{1}{\gamma!} \lambda_j \bar{\lambda}_k \mu_j^p \bar{\mu}_k^q B_j^\gamma \overline{B_k^\gamma}$$

$$= \sum_{p+q+r=s} \frac{s!}{p!\,q!} \sum_{|\gamma|=r} \frac{1}{\gamma!} \left(\sum_{j=1}^{N} \lambda_j \bar{\lambda}_k \mu_j^p B_j^\gamma\right) \overline{\left(\sum_{k=1}^{N} \lambda_k \mu_k^q B_k^\gamma\right)}.$$

We have $2p + r < s$ or $2q + r < s$ for p and q with $p + q + r = s$ and $p \neq q$. Thus, the terms in the aforementioned sum disappear whenever the induction hypothesis holds true. Consequently, we have

$$0 = \sum_{2p+r=s} \frac{s!}{p!\,p!} \sum_{|\gamma|=r} \frac{1}{\gamma!} \left| \sum_{j=1}^{N} \lambda_j \mu_j^p B_j^\gamma \right|^2.$$

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In order to complete the induction, we therefore infer that (38) is true when $2p + |\gamma| = s$.

Set $z^j := (\mu_j B_j)$. now. We can assume that all the pointsz$^1, \dots, z^N$ belong to $\mathbb{B}_{m+1}$. by scaling, if necessary. Then, we have by (38)

$$0 = \sum_{j=1}^{N} \lambda_j (z^j)^\gamma$$

for every multi-index γ The justification provided demonstrates that this amounts to

$$0 = \sum_{j,k=1}^{N} \lambda_j \overline{\lambda}_k \langle z^j, z^k \rangle^s$$

$s \geq 0$ for all integers. This suggests, together with (18),

$$0 = \sum_{j,k=1}^{N} \lambda_j \overline{\lambda}_k K^{m+2+\mathcal{E}}(z^j, z^k)$$

$\mathcal{E} \geq -2$,for any. In light of Lemma (3.2), we can deduce that $\lambda_1 = \cdots = \lambda_N = 0$as stated, is the case. The evidence is conclusive.

Corollary(3.4): Let a non-negative integer N, let $(\mu_1, B_1), \dots, (\mu_N, B_N)$ be distinct points in $\mathbb{C} \times \mathbb{C}^m$. Let $M > 0$ and $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. If

$$\sum_{j=1}^{N} \frac{\lambda_j \overline{\lambda}_k}{\left[1 - \zeta(\mu_j + \overline{\mu}_k \langle B_j, B_k \rangle)\right]^M} = 0 \qquad (22)$$

if all $\zeta \in \mathbb{C}$ are found close to the origin, then $\lambda_1 = \cdots = \lambda_N = 0$.

Proof: The sum on the left side of (22) is a holomorphic function of ζ near the origin. Therefore, the origin should be the point at which all Taylor coefficients vanish. It then follows

$$\sum_{j,k=1}^{N} \lambda_j \overline{\lambda}_k \left(\mu_j + \overline{\mu}_k \langle B_j, B_k \rangle\right)^s = 0$$

$s \geq 0$ for all integers. Thus, the corollary by Lemma is concluded (3.3).

We add new notation in this sentence. Let

$$\mathbb{B}_n^{(1)} := \{w \in \mathbb{B}_n : w_1 = 0\}.$$

Note that the slice in $\mathbb{B}_n$ traveling through $\boldsymbol{e}_n$ and w for $w \in \mathbb{B}_n^{(1)}$ is exactly the horodisk $\Delta_{|w|^{-2}}$, that is,

$$\Delta_{|w|^{-2}} = \{\lambda \in \mathbb{D} : \lambda \boldsymbol{e}_n + (1 - \lambda) w \in \mathbb{B}_n\}.$$

We write φ_w for the slice function denoted by given $w \in \mathbb{B}_n^{(1)}$ and fractional linear map $\varphi : \mathbb{B}_n \to \mathbb{D}$.

$$\varphi_w(\lambda) := \varphi(\lambda \boldsymbol{e}_n + (1 - \lambda) w), \ \lambda \in \Delta_{|w|^{-2}};$$

note $\varphi_w = \varphi$ when $n = 1$, because $w = 0$. Note that φ_w is holomorphic in a neighborhood of 1. Setting

$$G_\varphi(w) := \partial_1 \varphi(\boldsymbol{e}_n) - \sum_{j=2}^{n} \partial_j \varphi(\boldsymbol{e}_n) w_j \qquad (23)$$

and

$$H_\varphi(w) := \sum_{|\gamma|=2} \frac{\partial^\gamma \varphi(\boldsymbol{e}_n)}{\gamma!} (-1)^{\gamma_1} w_2^{\gamma_2} \dots w_n^{\gamma_n},$$

one may check that

$$\varphi_w(\lambda) = \varphi(\boldsymbol{e}_n) + G_\varphi(w)(\lambda - 1) + H_\varphi(w)(\lambda - 1)^2 + O(|\lambda - 1|^3) \qquad (24)$$

as $\lambda \to 1$ (uniformly in w).

When $\varphi(\boldsymbol{e}_n) = 1$ in addition, note from Lemma (2.4) that $G_\Phi(w)$ and $H_\varphi(w)$ reduce to

$$G_\varphi(w) = \partial_1 \varphi(\boldsymbol{e}_n) \qquad (25)$$

and

$$H_\varphi(w) = \frac{1}{2} \partial_{11} \varphi(\boldsymbol{e}_n) - \sum_{j=2}^{n} \partial_{1j} \varphi(\boldsymbol{e}_n) w_j. \qquad (26)$$

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When $n = 1$, the summations in (23) and (26) should be regarded as meaningless.

For a linear fractional map $\Phi : \mathbb{B}_n \to \mathbb{B}_m$ and $w \in \mathbb{B}_n^{(1)}$, we use the notation

$$\Phi_w := (\Phi_{1,w}, \dots, \Phi_{m,w}) \quad and \quad G_\Phi(w) := \left(G_{\Phi_1}(w), \dots, G_{\Phi_m}(w)\right) \qquad (27)$$

Remember that the j-th component function of is denoted by the symbol

$\Phi_{j,w} := (\Phi_j)_w$. The following lemma is entered:

$$L_{\Phi,\Psi}(t,w) := \left(1 + \frac{1}{t}\right)\partial_1\Phi_1(\boldsymbol{e}_n) + H_{\Phi_1}(w) + \overline{H_{\Psi_1}(w)} - \langle G_\Phi(w), G_\Psi(w)\rangle$$

for brevity.

Lemma(3.5): suppose that they are linear fractional maps $\Phi, \Psi : \mathbb{B}_n \to \mathbb{B}_m$,

$\Phi(\boldsymbol{e}_n) = \boldsymbol{e}_m$. Then the equality

$$\lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{1 - \langle \Phi_w(\lambda), \Psi_w(\lambda)\rangle}{|1-\lambda|^2} = \begin{cases} L_{\Phi,\Psi}(t,w) & \begin{array}{c} if\ \Phi(\boldsymbol{e}_n) = \Psi(\boldsymbol{e}_n)\ and \\ \partial_1\Phi_1(\boldsymbol{e}_n) = \partial_1\Psi_1(\boldsymbol{e}_n)\end{array} \\ \infty & otherwise \end{cases}$$

holds for $w \in \mathbb{B}_n^{(1)}$ and $0 < t < 1/|w|^2$.

Proof: Let $w \in \mathbb{B}_n^{(1)}$ and fix t such that $0 < t < 1/|w|^2$. Let $\varphi = \Phi_1$ and $\psi := \Psi_1$ for simplicity. Note

$$\frac{1 - \langle \Phi_w(\lambda), \Psi_w(\lambda)\rangle}{|1-\lambda|^2} = \frac{1 - |\varphi_w(\lambda)|^2}{|1-\lambda|^2} + \frac{\varphi_w(\lambda)\left[\overline{\varphi_w(\lambda)} - \overline{\psi_w(\lambda)}\right]}{|1-\lambda|^2} - \sum_{j=2}^{m} \frac{\Phi_{j,w}(\lambda)\Psi_{j,w}(\lambda)}{|1-\lambda|^2} \quad (28)$$

The boundaries of the three terms in the above right-hand side will be calculated separately. First, we determine the upper bound of the first term in the right-hand (28). Keep in mind that $\varphi_w : \Delta_{|w|^{-2}} \to \mathbb{D}$ is a fractional linear map when $\varphi_w(1) = 1$. As a result, according to Proposition (2.1), (24) and (25)

$$\frac{1 - |\varphi_w(\lambda)|^2}{|1-\varphi_w(\lambda)|^2} = \frac{1}{\varphi_w'(1)}\left\{1 + \frac{1}{t} + \frac{Re\,[\varphi_w''(1)]}{\varphi_w'(1)}\right\} - 1$$

$$= \frac{1}{\partial_1\varphi(\boldsymbol{e}_n)}\left\{1 + \frac{1}{t} + 2\frac{Re\left[H_\varphi(w)\right]}{\partial_1\varphi(\boldsymbol{e}_n)}\right\} - 1$$

for $\lambda \in \Gamma_t$, $\lambda \neq 1$. Note $\partial_1\varphi(\boldsymbol{e}_n) > 0$ by Lemma (2.4) (a). It follows that

$$\lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{|1-\varphi_w(\lambda)|^2}{|1-\lambda|^2} = \lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{|1-\varphi_w(\lambda)|^2}{|1-\varphi_w(\lambda)|^2} \cdot \frac{|1-\varphi_w(\lambda)|^2}{|1-\lambda|^2}$$

$$= \left(1 + \frac{1}{t}\right)\partial_1\varphi(\boldsymbol{e}_n) + 2Re\left[H_\varphi(w)\right] - [\partial_1\varphi(\boldsymbol{e}_n)]^2. \qquad (29)$$

Next, We determine the last term's limit in the right-hand side of (45).Since Φ_j, $2 \le j \le n$, is holomorphic in a neighborhood of 1 and $\Phi_j(\boldsymbol{e}_n) = 0$, we have by (24)

$$\Phi_{j,w}(\lambda) = G_{\Phi_j}(w)(\lambda - 1) + O(|\lambda - 1|^2).$$

The same holds for Ψ_j. It follows that

$$\lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{\Phi_{j,w}(\lambda)\overline{\Psi_{j,w}(\lambda)}}{|1-\lambda|^2} = G_{\Phi_j}(w)\overline{G_{\Psi_j}(w)} \qquad (30)$$

for each $j = 2, \dots, m$.

Finally, we determine the upper bound for the second term in the right-hand (28). If $\Phi(\boldsymbol{e}_n) \neq \Psi(\boldsymbol{e}_n)$, then it is evident that the limit being

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considered is ∞. So, assume $\Phi(\boldsymbol{e}_n) = \Psi(\boldsymbol{e}_n) = \boldsymbol{e}_m$. We have by (24) and (25)

$$\varphi_w(\lambda) - \psi_w(\lambda) = [\partial_1\varphi(\boldsymbol{e}_n) - \partial_1\psi(\boldsymbol{e}_n)](\lambda - 1) + [H_\varphi(w) - H_\psi(w)](\lambda - 1)^2 + O(|\lambda - 1|^3)$$

as $\lambda \to 1$. Thus we see that

$$\lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{|\varphi_w(\lambda) - \psi_w(\lambda)|}{|1-\lambda|^2} = \infty \quad if\ \partial_1\varphi(\boldsymbol{e}_n) \neq \partial_1\psi(\boldsymbol{e}_n).$$

The second portion of the lemma is proven by this. As opposed to that, if $\partial_1\varphi(\boldsymbol{e}_n) = \partial_1\psi(\boldsymbol{e}_n)$, then we obtain by (2) and (31)

$$\lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{\varphi_w(\lambda)\left[\overline{\varphi_w(\lambda)} - \overline{\psi_w(\lambda)}\right]}{|1-\lambda|^2} = \overline{H_\psi(w)} - \overline{H_\varphi(w)}. \qquad (32)$$

Also, note from (24)

$$[\partial_1\varphi(\boldsymbol{e}_n)]^2 = \partial_1\varphi(\boldsymbol{e}_n)\cdot\overline{\partial_1\psi(\boldsymbol{e}_n)} = G_\varphi(w)\overline{G_\psi(w)}.$$

Thus, by (29), (30), and we infer the first part of the lemma (32). The evidence is conclusive. For a linear fractional map $\Phi : \mathbb{B}_n \to \mathbb{B}_m$, recall that G_Φ denotes the map defined in (27).

Lemma(3.6): let a non-negative integer N, let $\Phi^1, \dots, \Phi^N : \mathbb{B}_n \to \mathbb{B}_m$ be distinct linear fractional maps such that

$$\Phi^1(\boldsymbol{e}_n) = \cdots = \Phi^N(\boldsymbol{e}_n) = \boldsymbol{e}_m \quad and \quad \partial_1\Phi_1^1(\boldsymbol{e}_n) = \cdots = \partial_1\Phi_1^N(\boldsymbol{e}_n). \qquad (33)$$

Then, the following claims are true:

(a) When $n = 1$, the vectors

$$\left(H_{\Phi_1^j}(0), G_{\Phi^j}(0)\right) \in \mathbb{C}\times\mathbb{C}^m,\ \ j = 1, \dots, N$$

are all distinct;

(b) When $n \geq 2$, the vectors

$$\left(H_{\Phi_1^j}(w), G_{\Phi^j}(w)\right) \in \mathbb{C}\times\mathbb{C}^m,\ \ j = 1, \dots, N$$

are all distinct for almost every $w \in \mathbb{B}_n^{(1)}$.

(31) The (n-1)-dimensional volume measure on $w \in \mathbb{B}_n^{(1)} \cong \mathbb{B}_{n-1}$ is what is meant by "nearly every" in this context $(n-1)$.

Proof: From Proposition, Assertion (a) immediately follows (2.5). So, for the remainder of the proof, assume $n \geq 2$. Let $w \in \mathbb{B}_n^{(1)}$. Since we have (33) by assumption, we note from (25) that $G_{\Phi_1^j}(w) = \partial_1\Phi_1^j(\boldsymbol{e}_n)$ are all the same. So, by (26), we only need to consider the vectors

$$Q_j(w) := \left(\frac{1}{2}\partial_1\Phi_1^j(\boldsymbol{e}_n) - \sum_{k=2}^{n}\partial_{1,k}\Phi_1^j(\boldsymbol{e}_n)w_k, G_{\Phi_2^j}(w), \dots, G_{\Phi_m^j}(w)\right)$$

for $j = 1, \dots, N$. For each j, setting

$$A_j := \left(\frac{\partial_{11}\Phi_1^j}{2}, \partial_1\Phi_2^j, \dots, \partial_1\Phi_m^j\right)$$

and

$$M_j := \begin{pmatrix} \partial_{12}\Phi_1^j & \cdots & \partial_{1n}\Phi_1^j \\ \partial_2\Phi_2^j & \cdots & \partial_n\Phi_2^j \\ \vdots & \vdots & \vdots \\ \partial_2\Phi_m^j & \cdots & \partial_n\Phi_m^j \end{pmatrix}_{m\times(n-1)},$$

note

$$Q_j(w) = M_j(\boldsymbol{e}_n)w' + A_j(\boldsymbol{e}_n)$$

where $w' := (w_2, \dots, w_n)$ and the matrix $M_j(\boldsymbol{e}_n)$ is regarded as a linear operator from $\mathbb{C}^{n-1}$ to $\mathbb{C}^m$. Since $\Phi^1, \dots, \Phi^N$ are distinct maps satisfying (33) by assumption, note from Proposition (2.5) that

$$either\ A_j(\boldsymbol{e}_n) \neq A_k(\boldsymbol{e}_n) \quad or \quad M_j(\boldsymbol{e}_n) \neq M_k(\boldsymbol{e}_n)$$

whenever $j \neq k$. Thus, if $M_j(\boldsymbol{e}_n) = M_k(\boldsymbol{e}_n)$ and $j \neq k$, then we see

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$$Q_j(w) - Q_k(w) = A_j(\boldsymbol{e}_n) - A_k(\boldsymbol{e}_n) \neq 0$$

for all w. So, for simplicity, we may assume $M_j(\boldsymbol{e}_n) \neq M_k(\boldsymbol{e}_n)$ whenever $j \neq k$. Given $j < k$, let S_{jk} be the set of all $\zeta \in \mathbb{C}^{n-1}$ such that

$$[M_j(\boldsymbol{e}_n) - M_k(\boldsymbol{e}_n)] = A_k(\boldsymbol{e}_n) - A_j(\boldsymbol{e}_n).$$

Note that S_{jk} lies in a hyperspace of $\mathbb{C}^{n-1}$, because the kernel of

$M_j(\boldsymbol{e}_n) - M_k(\boldsymbol{e}_n) \neq 0$ cannot be of full dimension. In particular, the set

$$S := \bigcup_{1 \le j < k \le N} S_{jk},$$

is a part of the measure 0 subset of $\mathbb{C}^{n-1}$. We get the lemma's conclusion because $Q_1(w), \dots, Q_N(w)$ are all distinct if and only if $w' \notin S$. The evidence is conclusive.

We are now prepared to demonstrate our key finding. Thesis (1.1). We shall make advantage of the fact that a bounded linear operator is closed and bounded if and only if its adjoint is closed and bounded when it leaves one Hilbert space and enters another. Along with this observation, we make note of a straightforward but crucial fact regarding adjoints of composition operators. When $C_\Phi : A^2_{1+\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+\mathcal{E}}(\mathbb{B}_n)$ is bounded, its adjoint $C^*_\Phi : A^2_{1+2\mathcal{E}}(\mathbb{B}_n) \to A^2_{1+\mathcal{E}}(\mathbb{B}_m)$ takes the practice of reproducing kernels to a level where

$$C^*_\Phi K^{n,1+2\mathcal{E}}_z = K^{m,1+\mathcal{E}}_{\Phi(z)}, \tag{34}$$

which the replicating property readily verifies.

For convenience, we repeat Theorem (1.1) in the manner that follows.

Theorem (3.7) Given a positive integer N, let $\Phi^1, \dots, \Phi^N : \mathbb{B}_n \to \mathbb{B}_m$ be distinct linear fractional maps such that $\|\Phi^j\|_\infty = 1$ for $j = 1, \dots, N$. For $\mathcal{E} \ge -1$ with

$m = n + \mathcal{E}$ and $\lambda_1, \dots, \lambda_N \in \mathbb{C}$, assume that

$\sum_{j=1}^N \lambda_j C_{\Phi^j} : A^2_{1+2\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+\mathcal{E}}(\mathbb{B}_n)$ is compact. Then $\lambda_1 = \dots = \lambda_N = 0$.

Proof: First, we show

$$\lim_{|z|\to 1^-} \sum_{j,k=1}^N \lambda_j \bar{\lambda}_k \left(\frac{1 - |z|^2}{1 - \langle \Phi^j(z), \Phi^k(z) \rangle} \right)^{n+2+2\mathcal{E}} = 0. \tag{35}$$

Put $T := \sum_{j=1}^N \lambda_j C_{\Phi^j}$ for short. Since $T : A^2_{1+\mathcal{E}}(\mathbb{B}_m) \to A^2_{1+2\mathcal{E}}(\mathbb{B}_n)$ is compact by assumption, its adjoint $T = \sum_{k=1}^N \bar{\lambda}_k C^*_{\Phi^k} : A^2_{1+2\mathcal{E}}(\mathbb{B}_n) \to A^2_{1+\mathcal{E}}(\mathbb{B}_m)$ is also compact. Note from (18) and (19) that, as $|z| \to 1^-$, the normalized kernel $K^{n,1+2\mathcal{E}}_z / \|K^{n,1+2\mathcal{E}}_z\|_{n,1+2\mathcal{E}}$ converges to 0 uniformly on closed and bounded sets in $\mathbb{B}_n$, or equivalently, converges to 0 weakly in $A^2_{1+2\mathcal{E}}(\mathbb{B}_n)$. It follows that since a compact operator maps a weakly convergent sequence onto a norm convergent one.

$$T^* \left(\frac{K^{n,1+2\mathcal{E}}_z}{\|K^{n,1+2\mathcal{E}}_z\|_{n,1+2\mathcal{E}}} \right) \to 0 \quad in \quad A^2_{1+\mathcal{E}}(\mathbb{B}_m) \tag{36}$$

as $|z| \to 1^-$. Meanwhile, for $z \in \mathbb{B}_n$, we have by (34)

$$\begin{aligned} \|T^* K^{n,1+2\mathcal{E}}_z\|_{m,1+\mathcal{E}} &= \sum_{j,k=1}^N \lambda_j \bar{\lambda}_k \langle C^*_{\Phi^k} K^{n,1+2\mathcal{E}}_z, C^*_{\Phi^j} K^{n,1+2\mathcal{E}}_z \rangle_{A^2_{1+\mathcal{E}}(\mathbb{B}_m)} \\ &= \sum_{j,k=1}^N \lambda_j \bar{\lambda}_k \langle K^{m,1+\mathcal{E}}_{\Phi^k(z)}, K^{m,1+\mathcal{E}}_{\Phi^j(z)} \rangle_{A^2_{1+\mathcal{E}}(\mathbb{B}_m)} \\ &= \sum_{j,k=1}^N \lambda_j \bar{\lambda}_k K^{m,1+\mathcal{E}}_{\Phi^k(z)} \left(\Phi^j(z) \right) \end{aligned}$$

so that

$$\left\| T^* \frac{K^{n,1+2\mathcal{E}}_z}{\|K^{n,1+2\mathcal{E}}_z\|_{n,1+2\mathcal{E}}} \right\|^2_{m,1+\mathcal{E}} = \sum_{j,k=1}^N \lambda_j \bar{\lambda}_k \left(\frac{1 - |z|^2}{1 - \langle \Phi^j(z), \Phi^k(z) \rangle} \right)^{n+2+2\mathcal{E}} \tag{37}$$

For this equality, we used (18), (19), and the relationship $m = n + \mathcal{E}$. Therefore, we have (35) by (36) and (37).

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We now demonstrate $\_\lambda_1 = 0$, and one may demonstrate $\lambda_2 = \cdots = \lambda_N = 0$. by the exact same justification. Since it is assumed that $\|\Phi^1\|_\infty = 1$, there are some $\zeta \in \mathbb{S}_n$ and $\eta \in \mathbb{S}_m$ such that $\Phi^1(\zeta) = \eta$. After unitary variable changes, we can infer that $\zeta = \boldsymbol{e}_n$ and $\eta = \boldsymbol{e}_m$, resulting in $\Phi^1(\boldsymbol{e}_n) = \boldsymbol{e}_m$. J, the index set provided by

$$J := \{j : \Phi^j(\boldsymbol{e}_n) = \Phi^1(\boldsymbol{e}_n) \ \ and \ \partial_1\Phi_1^j(\boldsymbol{e}_n) = \partial_1\Phi_1^1(\boldsymbol{e}_n)\}.$$

Using Lemma (3.6), pick $w \in \mathbb{B}_n^{(1)}$ such that

$$\left(H_{\Phi_1^j}(w), G_{\Phi^j}(w)\right) \neq \left(H_{\Phi_1^k}(w), G_{\Phi^k}(w)\right) \quad for \ \ j,k \in J \ \ with \ \ j \neq k. \quad (38)$$

Let $0 < t < 1/|w|^2$. Applying (35) along the curve $z_\lambda := \lambda_{e_n} + (1-\lambda)w$ for $\lambda \in \Gamma_t$ and then using $1 - |z_\lambda|^2 = (1/t - |w|^2)|1-\lambda|^2$, we obtain

$$0 = \lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \sum_{j,k=1}^{N} \lambda_j\bar{\lambda}_k \left(\frac{|1-\lambda|^2}{1 - \langle \Phi_w^j(\lambda), \Phi_w^k(\lambda)\rangle}\right)^{n+2+2\varepsilon} = \lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \left\{\sum_{j,k\in J} + \sum_{j,k\notin J}\right\}$$

where the source of the second equality is Lemma (3.5). Additionally, each term in the brace of the previous statement is rendered nonnegative by (37) and disappears. Lemma (3.5) again suggests that

$$0 = \lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \sum_{j,k\in J} \lambda_j\bar{\lambda}_k \left(\frac{|1-\lambda|^2}{1 - \langle \Phi_w^j(\lambda), \Phi_w^k(\lambda)\rangle}\right)^{n+2+2\varepsilon}$$

$$= \sum_{j,k\in J} \lambda_j\bar{\lambda}_k \left(\lim_{\substack{\lambda\to 1\\ \lambda\in\Gamma_t}} \frac{|1-\lambda|^2}{1 - \langle \Phi_w^j(\lambda), \Phi_w^k(\lambda)\rangle}\right)^{n+2+2\varepsilon}$$

$$= \sum_{j,k\in J} \frac{\lambda_j\bar{\lambda}_k}{\left[L_{\Phi^j,\Phi^k}(t,w)\right]^{n+2+2\varepsilon}}.$$

Note_$\delta := \partial_1\Phi_1^1(\mathbf{e}_n) > 0$ by Lemma (2.4) (a). So, multiplying both sides of the above by $\left(\frac{t}{\delta}\right)^{-(n+2+2\varepsilon)}$,We succeed

$$\sum_{j,k\in J} \frac{\lambda_j\bar{\lambda}_k}{\left[\frac{t}{\delta}L_{\Phi^j,\Phi^k}(t,w)\right]^{n+2+2\varepsilon}} = 0.$$

For any $j,k \in J$, Observe that the purpose

$$t \longmapsto \frac{t}{\delta}L_{\Phi^j,\Phi^k}(t,w) \quad (w : \text{fixed})$$

$$= 1 - t\left[\frac{\langle G_{\Phi^j}(w), G_{\Phi^k}(w)\rangle - H_{\Phi_1^j}(w) - \overline{H_{\Phi_1^k}(w)}}{\delta} - 1\right]$$

extends unquestionably to a holomorphic function in the area around the origin. Thus, from (38) and Corollary (3.4), we deduce that $\lambda_j = 0$ for all $j \in J$. In particular, we reach the necessary conclusion that $\lambda_1 = 0$. The evidence is conclusive.

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REFERENCES

[1] B. R. Choe, H. Koo and I. Park, Compact differences of composition operators over polydisks, Integral Equations Operator Theory 73(1)(2012), 57–91.
[2] B. R. Choe, H. Koo and I. Park, Compact differences of composition operators on the Bergman spaces over the ball, Potential Analysis 40(2014), 81–102.
[3] C. C. Cowen and B. D. MacCluer, Composition operators on spaces of analytic fuctions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.
[4] C. C. Cowen and B. D. MacCluer, Linear fractional maps of the ball and their composition operators, ActaSci. Math. (Szeged) 66(1-2)(2000), 351–376.
[5] E. Gallardo-Guti´errez, M. Gonz´alez, P. Nieminen and E. Saksman, On the connected component of compact composition operators on the Hardy space, Adv. Math. 219(2008), 986–1001.

© 2021 JPPW. All rights reserved

[6] K. Heller, B. D. MacCluer and R. J. Weir, Compact differences of composition operators in several variables,Integral Equations Operator Theory 69(2)(2011), 247-268.

[7] H. Hunziker, H. Jarchow and V. Mascioni, Some topologies on the space of analytic self-maps of the unit disk, Geometry of Banach spaces (Strobl, 1989), 133–148, London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press, Cambridge, 1990.

[8] L. Jiang and C. Ouyang, Compact differences of composition operators on holomorphic function spaces in the unit ball, Acta Math. Sci. Ser. B Engl. Ed. 31(5)(2011), 1679–1693.

[9] H. Koo and W. Smith, Composition operators induced by smooth self-maps of the unit ball in CN, J. Math. Anal. Appl. 329(2007), 617-633.

[10] H. Koo and M. Wang, Joint Carleson measure and the difference of composition operators on $A_{\alpha}^{p}(\mathbb{B}_{m})$, J. Math. Anal. Appl. 419(2014), 1119–1142.

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[11] T. Kriete and J. Moorhouse, Linear relations in the Calkin algebra for composition operators, Trans. Amer. Math. Soc. 359(6)(2007), 2915-2944.

[12] B. D. MacCluer, Components in the space of composition operators, Integral Equations Operator Theory 12(5) (1989), 725-738.

[13] J. Moorhouse, Compact differences of composition operators, J. Funct. Anal. 219(2005), 70–92.

[14] P. J. Nieminen and E. Saksman, On compactness of the difference of composition operators, J. Math. Anal. Appl. 298(2004), 501–522.

[15] E. Saukko, Difference of composition operators between standard weighted Bergman spaces, J. Math. Anal. Appl. 381(2011), 789–798.

[16] E. Saukko, An application of atomic decomposition in Bergman spaces to the study of differences of composition operators, J. Funct. Anal. 262(2012), 3872–3890.

[17] J. H. Shapiro, Composition operators and classical function theory, Springer, New York, 1993.

[18] J. H. Shapiro and C. Sundberg, Isolation amongst the composition operators, Pacific J. Math. 145(1990),117–152.

[19] K. Zhu, Spaces of holomorphic functions in the unit ball, Springer-Verlag, New York, 2005.

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